Coupled Pendula

D.S. Mackey, F. De Terán, R. Vandebril, I. Bossuyt, A. Herremans, N. Scheerlinck, H. Vanaenroyde

KU Leuven, Universidad Carlos III de Madrid, Western Michigan University

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Outline

Example of the setup

Question of the day

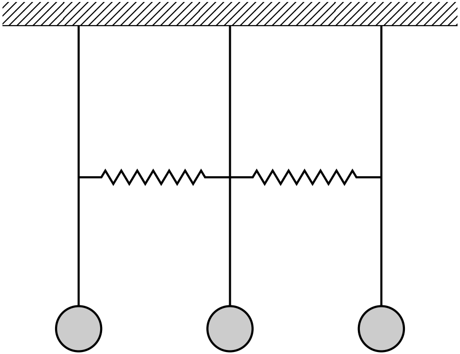
Can linear algebra help us to predict the movement of the coupled pendula?

Start

- Fetch your material.
- Make your pendula: attach the main rope to a table that is turned upside down.
- Attention:
  - Symmetry is important!
  - All ropes attached to the balls should be equally long.
- The balls must swing perpendicular to the rope.
- Move 1 ball, move 2 balls.
- It is clearly a coupled system.

Outline

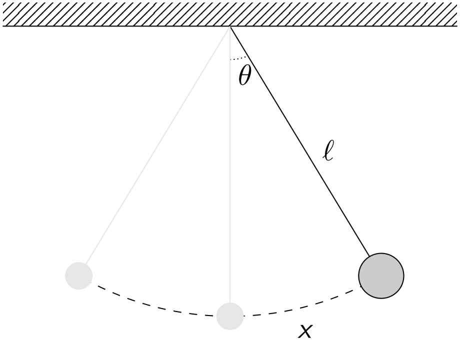
Mathematical model: three coupled pendula



Assumptions

For simplicity we assume the movement of the pendula in the plane (right to left), instead of perpendicular to the plane as in our setup.

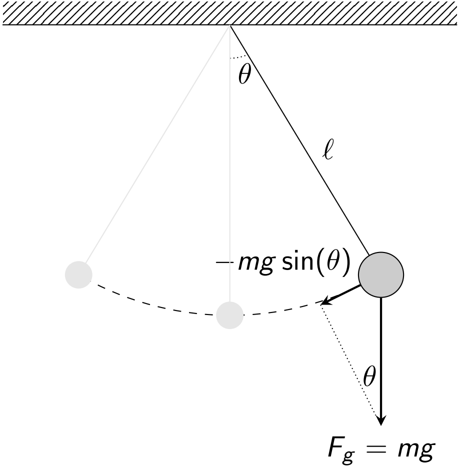
Single pendulum



Excitement angle

- The arclength $x = \theta\ell$.
- The angle $\theta = \frac{x}{\ell}$.

Single pendulum



Differential equation

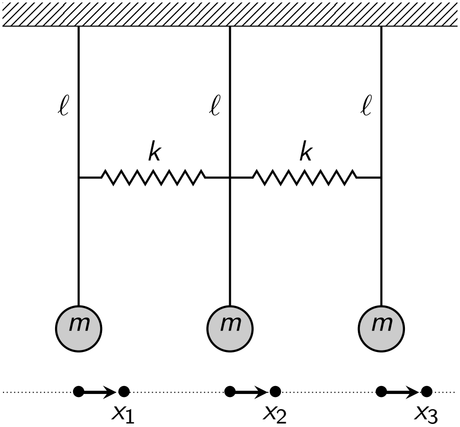
We have

- $F = ma = -mg\sin(\theta)$
- $m\ddot{x} = -mg\sin(\theta)$
- $m\ddot{x} \approx -mg\theta \quad (\text{small } \theta)$
- $m\ddot{x} \approx -mg\left(\frac{x}{\ell}\right)$

Gravity leads to:

$$m\ddot{x} \approx -\frac{mg}{\ell}x.$$

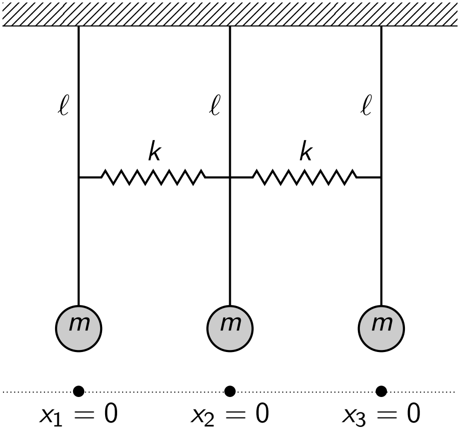
Mathematical model: three coupled pendula



Displacement

The unknowns x_1, x_2, and x_3 represent the displacement from the equilibrium.

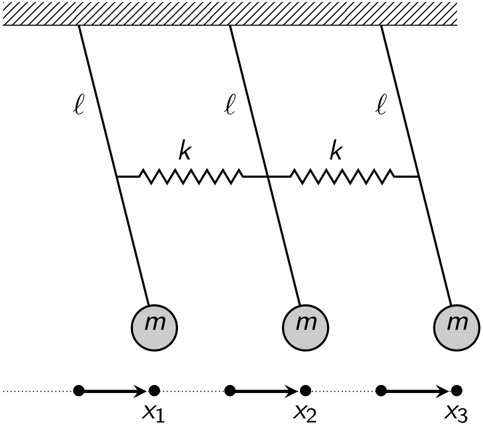
Mathematical model: three coupled pendula



System at Rest

$x_1 = 0$, $x_2 = 0$, and $x_3 = 0$.

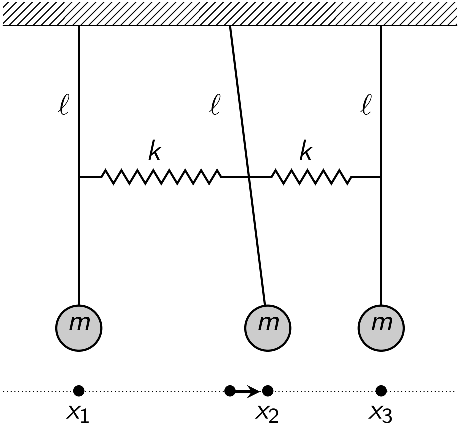
Mathematical model: three coupled pendula



Equal excitement

System having $x_1 = x_2 = x_3 > 0$.
The springs are unstressed, they are not under tension.

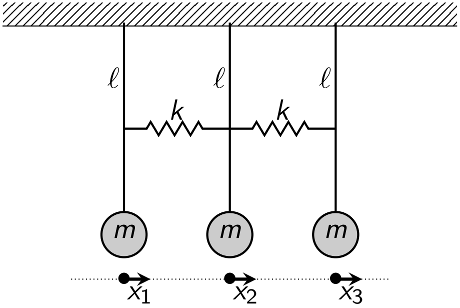
Mathematical model: three coupled pendula



General setting

Left spring is stretched, since $x_2 - x_1 > 0$.
Right spring is compressed, since $x_3 - x_2 < 0$.

Mathematical model: three coupled pendula



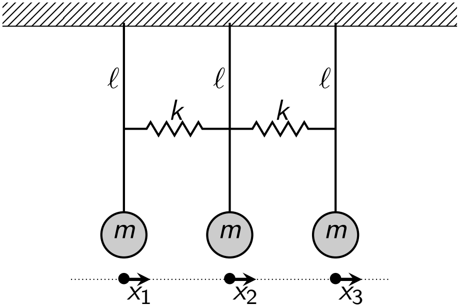
Force on a single spring

$$F = kx.$$

Forces on a spring

- $x_2 - x_1 = 0$ spring is not stretched, not compressed.
- $x_2 - x_1 > 0$ spring is stretched.
- $x_2 - x_1 < 0$ spring is compressed.

Coupled pendula



Coupled equations

$$
\begin{cases}
m\ddot{x}_1 = -\frac{mg}{\ell}x_1 + k(x_2 - x_1) \\
m\ddot{x}_2 = -k(x_2 - x_1) - \frac{mg}{\ell}x_2 + k(x_3 - x_2) \\
m\ddot{x}_3 = -k(x_3 - x_2) - \frac{mg}{\ell}x_3
\end{cases}
$$

Coupled pendula

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Matrix version

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} = -\begin{bmatrix} \frac{g}{\ell} + \frac{k}{m} & -\frac{k}{m} & 0 \\ -\frac{k}{m} & \frac{g}{\ell} + 2\frac{k}{m} & -\frac{k}{m} \\ 0 & -\frac{k}{m} & \frac{g}{\ell} + \frac{k}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Matrix version

Simplified writing

$$
\begin{aligned}
\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} &= -\begin{bmatrix} \frac{g}{\ell} + \frac{k}{m} & -\frac{k}{m} & 0 \\ -\frac{k}{m} & \frac{g}{\ell} + 2\frac{k}{m} & -\frac{k}{m} \\ 0 & -\frac{k}{m} & \frac{g}{\ell} + \frac{k}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\
&= -\begin{bmatrix} a+b & -b & 0 \\ -b & a+2b & -b \\ 0 & -b & a+b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \\
\ddot{\mathrm{x}} &= -K\mathrm{x}.
\end{aligned}
$$

We use the variables $a = \frac{g}{\ell}$ and $b = \frac{k}{m}$.

Outline

What is a normal mode

Normal Mode

A normal mode of a system is a motion in which all the particles *execute simple harmonic motion (e.g. a nice cosine) with* the same frequency *and* the same phase.

Note, however, that the amplitude of the particles can be different.

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- what does the same frequency mean:

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Questions: What are in our setting

- moving particles: balls on the pendulum,
- what does the same frequency mean: moving equally fast,
- what does the same phase mean: pass through the origin together.

Normal modes

Assignment: Set 1 ball in motion

Is this a normal mode, that is:

- Do all particles have the same frequency?
- Do the particles pass through the origin simultaneously?
- Do all particles make the same simple harmonic motion?

Normal modes

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Assignment

- Find one normal mode!
- Can you find another one?

Relation between physics and linear algebra

Expression for a normal Mode

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} a_1 \cos(\beta t + \gamma) \\ a_2 \cos(\beta t + \gamma) \\ a_3 \cos(\beta t + \gamma) \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \cos(\beta t + \gamma).$$

Vector notation gives us: $\mathrm{x} = \mathrm{a} \cos(\beta t + \gamma)$.

Does this satisfy the requirements?

- Simple harmonic motion?
- Same frequency?
- Same phase?
- Can the amplitude be different?

Relation between physics and linear algebra

System to be solved

Substituting the expression

$$x = a\cos(\beta t + \gamma)$$

into

$$\ddot{x} = -Kx$$

yields the following system that we have to solve:

$$-\beta^2 a\cos(\beta t + \gamma) = -Ka\cos(\beta t + \gamma).$$

Or simplified:

$$\beta^2 a = Ka \quad \longrightarrow \quad Ka = \beta^2 a.$$

Relation between physics and linear algebra

What kind of problem is this?

For a matrix K, we therefore look for a scalar β, and a vector a such that:

$$K\mathsf{a} = \beta^2 \mathsf{a}.$$

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Eigenvalue problem

- So we look for the eigenvectors and eigenvalues of K.
- The eigenvalue is related to the frequency.
- The eigenvector represents the amplitudes at time 0 (initial position).
  - The ratio between the displacements of each pendulum.
  - If you start with an eigenvector as displacement, you are in an eigenmode.

Assignment

Questions

- Find normal modes (eigenvectors) using your intuition.
- How many normal modes will your system have?
  (This is equal to the number of eigenvectors.)
- Test whether the vector of amplitudes of your found normal modes is an eigenvector of K.

Properties via matrix structure

What structure does our matrix have

$$\begin{bmatrix} a+b & -b & 0 \\ -b & a+2b & -b \\ 0 & -b & a+b \end{bmatrix}$$

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- Eigenvalues:

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- Eigenvectors form a basis.
- Eigenvectors:

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Symmetric matrix – What do you know?

- Eigenvalues: Real.
- Eigenvectors form a basis.
- Eigenvectors: Are orthogonal.
  (If you have two normal modes, the third one must be perpendicular to these.)

All eigenmodes

Exercise

- Find all modes!
- Use the orthogonality property if you have already found 2.

Solution

Eigenvector 1

$$\begin{bmatrix} a+b & -b & 0 \\ -b & a+2b & -b \\ 0 & -b & a+b \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

with the first eigenvalue $\lambda_1 = a = g/\ell$.

Solutions

Eigenvector 2

$$\begin{bmatrix} a+b & -b & 0 \\ -b & a+2b & -b \\ 0 & -b & a+b \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = (a+b) \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

with $\lambda_2 = a + b$ and $a = g/\ell$ and $b = k/m$.

Solutions

Eigenvector 3

$$\begin{bmatrix} a+b & -b & 0 \\ -b & a+2b & -b \\ 0 & -b & a+b \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = (a+3b) \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

with $\lambda_3 = a + 3b$, where $a = g/\ell$ and $b = k/m$.

Solutions

Orthogonal eigenvectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Extra – Compute the length ℓ

Calculate ℓ

- Determine the frequency of the normal mode corresponding to the eigenvector $[1, 1, 1]^T$.
- Use this frequency to find ℓ.
- Tip: the frequency and the eigenvalues are linked!
  - The frequency $f = \frac{\sqrt{\lambda}}{2\pi}$, with λ the eigenvalue.
  - For eigenvector $[1, 1, 1]^T$ we have $\lambda_1 = a = g/\ell$.
- Remark: the lenght ℓ is measured from the rope to the *center* of the bob!

Extra – Pendulum with 4 bobs

Make a pendulum with 4 bobs

- Find the eigenmodes.
- Do this mathematically, by computing the eigenvectors of the matrix.

Solutions: pendulum with 4 bobs

Eigenvalue problem

The matrix K becomes:

$$\begin{bmatrix} a+b & -b & 0 & 0 \\ -b & a+2b & -b & \\ 0 & -b & a+2b & -b \\ 0 & 0 & -b & a+b \end{bmatrix}.$$

The eigenvectors are:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1+\sqrt{2} \\ 1 \\ -1 \\ -1-\sqrt{2} \end{bmatrix} \qquad \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ -1-\sqrt{2} \\ 1+\sqrt{2} \\ -1 \end{bmatrix}$$