Thirty Annotations with Further Commentary on the Coupled Pendula Demonstration Notes, and a new Appendix

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Abstract

These thirty annotations are keyed to locations in the handwritten notes, and constitute some additional commentary on issues connected with the coupled pendula "toy". The new Appendix discusses the "beats" that are observable in the toy, as well as the behavior of analogous systems with more than three pendula.

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NOTE #

(1) History: Nov 1998 is the first time that I actually prepared some detailed, extended notes for this presentation, since I was doing it as a guest lecture in a sophomore level first linear algebra course at Kalamazoo College. (And then the following week as a guest lecture in a first semester graduate class on Applied Linear Algebra at Western Michigan.) So I wanted to try to organize it so that I would not forget anything important.

Prior to this I would just do things from memory and some very short and basic notes, writing on the blackboard as I went. I first started using a much shorter version of this talk when I was a graduate student doing recitation sessions, some time in the mid-to-late 1980's; I don't remember exactly when.

(2) These first two or three slides (transparencies) go faster or slower, depending on the audience, and the stage of the course that we are at. In a course on DEs with mainly engineering students, this goes pretty fast, since they will have seen most of these ideas already. In a linear algebra class, this often needs a little longer. I just play it by ear, and so it is different for each audience.

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(3) Notation: These numbers refer to the old-fashioned "transparency" slides sprinkled throughout the video. I.e., the note labelled n refers to slide n, with the number in the upper right corner.

(4) Missing forces: There are actually two forces that are missing from the force diagram on this slide. They are the radial component of the gravitational force, and the tension in the suspension string that exactly balances that radial component. They have been left out for visual clarity, but should probably be mentioned for the sake of completeness.

(5) Setting up the model: Here is where I set up a model for the pictured system, where the pendula all move in the plane of the paper, rather than in parallel planes perpendicular to the paper, as in our actual device. I would do this with the green circles on the slide starting out to be empty, and then fill them in as we discussed the effect of the spring forces on each ball. Some emphasis is placed on getting students to see clearly why the plus/minus signs in these coupling terms are the way that they are.

For this system, the fact that coupling forces act only on nearest neighbors is physically evident, and explains why the variable x_3 does not appear in the first eqn, and x_1 does not appear in the third eqn. And as a consequence, the matrix K in the next slide has zero entries in the $(1,3)$ and $(3,1)$ entries. For our toy, the restriction to only nearest neighbor coupling is a debateable assumption, but perhaps can be tentatively justified as ignoring "neligible" effects. One can then examine and evaluate this assumption "after the fact" in terms of the coherence between the model's predictions and the actual observed behavior of the toy.

In actual practice, this issue rarely arises with students, who are usually overly willing to accept things without much critical pushback. But when they do raise this issue, it is good to defer it to an outside-the-class discussion, and use it as an opportunity to discuss some of the more philosophical issues around mathematical modeling of physical phenomena (i.e., sometimes more of an art than a science?).

(6) Plus/minus signs: Why is this sign plus in the first equation, but instead is minus in the second equation? This is an issue that I think is worth getting clear with students, since getting these signs right makes a huge difference in the properties of the matrix K that shows up in the next slide.

(7) Hand-waving claim: To me, this is the only problematic part of the talk, and I usually try to gloss over this as fast as possible, with a certain amount of "hand-waving". Here I am setting up a model for a physical system that is clearly not identical to the one I have demonstrated at the beginning of the talk, but yet I am claiming that it is also a "good" model for the coupled pendula. For young students, there is usually very little resistance to this assertion, since it is coming from "the professor" (the authority figure). But once in a while a student will not be satisfied with my hand-waving, so I postpone that discussion to outside the classroom, with just the student(s) who raised concerns.

This does not have to be a bad thing, and can actually be an opportunity(!) to talk about the making of mathematical models for physical systems more generally. For example, topics could include: the use of approximations and simplifications, the focusing in on what seem to be the essential features of an otherwise complicated scenario, the a posteriori justification of a model by the correspondence (or lack of same) between the predictions made by a model and the observed physical reality, ... etc.

Indeed, this device might well be used in a course on mathematical modelling, and not just in a linear algebra or differential equations course.

(8) The “stiffness” matrix K: This is of course just a reorganization of the previous slide into matrix form, but even here there are some interesting features that students should be sure to take notice of. The first form of the matrix version (in black) still looks rather complicated, but then observe that it is not the individual parameters g, ℓ, k, m that matter, but only the ratios g/ℓ and k/m that ever appear. And so simplifying to take that into account gives the easier-to-handle version with the matrix K in green.

Now of course I ask the students if they notice anything special about the matrix K. Usually (but not always) somebody will say pretty quickly that the matrix is symmetric. I ask the students to wonder a little bit about this, since for beginning students, the property of a matrix being symmetric often just seems like something artificial and made up, not something that is ever likely to appear in a ”real” application. (Certainly that is what I thought when I was an undergraduate student.) And yet here it is, symmetry emerging almost as if by magic. And looking back at the original formulation on slide 2 makes it seem (at least to me) even more unlikely that this K should end up being symmetric. In fact, a quick look at the system on slide 2 might make you even expect a skew-symmetric matrix to appear for K (or at least the off-diagonal part to be skew-symmetric).

At this point I usually leave it a bit of a mystery as to ”why” the matrix K comes out to be symmetric, but I do emphasize that, despite what you might expect, the appearance of symmetric matrices is extremely common in applications. And that is why any theorems about the properties of symmetric matrices are especially valuable and important.

(9) Gerschgorin: This note is something I added in more recent years. I have never discussed this in a first undergraduate DE or linear algebra course. However, I have mentioned it several times in a second undergrad linear algebra or 1st-semester grad linear algebra course. The Gerschgorin theorem immediately gives simple bounds on the eigenvalues, and also the qualitatively important information (for this physical application) that the eigenvalues must all be positive. These bounds can be confirmed later, once we know what the eigenvalues actually turn out to be. This also gives some early validation to the physical plausibility of the model.

And this also gives an opportunity to mention (or discuss more fully if desired) the importance of positive definite matrices for applications, especially for vibration problems. Of course all of this can be delayed until later in the presentation, too, or just left out completely.

(10) Normal modes: I think it is important for students (especially engineering students) to start out with a “physical” definition of normal mode, one that they can work with and

think about in terms of concepts that they are already familiar with, rather than just a purely mathematical one. And then the mathematical definition can come later. Here the phrase "same phase" probably needs some clarification, since it is easy to misunderstand it to mean that all of the balls are on the "same side" all of the time. What it is intended to indicate is that all of the balls pass through their respective equilibrium points simultaneously, but that they are allowed to approach those equilibrium points from "opposite sides". In addition, particles have the "same phase" if they all reach their maximum displacements from their respective equilibrium points simultaneously, even if those displacements are on "opposite sides". Information about the balls being on the same or on opposite sides is encoded in the "amplitude pattern vector", to be introduced in the next slide.

It may be worth emphasizing this property of "simultaneous-passage-through-equilibrium-points" as one key indicator when confirming that a motion of the system is indeed a normal mode.

(11) Normal mode ↔ eigenstuff connection: Here in these slides (5 and 6) is where we see the first big payoff of this discussion of the toy: the connection between the physical concept of normal mode with the linear algebra concepts of eigenvector and eigenvalue.

First normal mode gets translated into mathematical language, with the introduction of the additional physical notion of "amplitude pattern vector". Then connecting that to the DE system leads to linear algebra!

And finally we see why it is physically important to have positive eigenvalues. It is not just some kind of mathematical convenience, but a physical *necessity*. And any model that does not produce them cannot be a good model.

(12) Gerschgorin again: Here is an alternative spot to bring in Gerschgorin, if you want. Now that we know that the positivity of eigenvalues is important, the information obtainable from Gerschgorin is especially relevant.

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NOTE #

(13) Eigenvector excitation (a): There are now various options for finding the normal modes of the toy. One is to just try and guess them from physical intuition, and then check whether they are correct. Two kinds of check are possible here, one mathematical (for a guess v, compute Kv and see if it is a multiple of v or not), the other physical (start off the toy system in the amplitude pattern denoted by v, and see if you get simple harmonic motion with the same frequency for all three balls, as well as simultaneous-passage-through-equilibria).

Another option is to try to just grind out the eigenvectors and eigenvalues by brute force calculation of the characteristic polynomial (in terms of the parameters a and b), etc... The students can see that this will be pretty tedious and are not enthusiastic about this approach, although it is actually not too bad in the end.

(In the book by H.V.McIntosh, where I first discovered this toy, he does exactly this tedious calculation of the eigenvalues from the characteristic polynomial, including all of the symbolic parameters. Then instead of finding the eigenvectors in the usual undergraduate way of solving the linear system $Kv = \lambda v$, he finds the eigenvectors by a more unexpected pathway, namely by calculating the spectral projectors(!), via explicit formulas. In this case these projectors are all rank-one matrices, so any column of one of these projectors will be an eigenvector.)

Another option is to plug in plausible numerical values for a and b, and calculate numerically. Now $a = (g/\ell)$ is a physically accessible number, since you can easily measure the length ℓ. But $b = (k/m)$ is not so easy; m is measurable, but it is hard to see how to get your hands on a sensible value for k. In any case, I think there is a strong case for not going to numerical values right away like this. I will say more about this in later comments.

(Continue in next note.)

(14) Eigenvector excitation (b): I usually start with asking students to guess, and then try to check physically. Sometimes students are able to guess all three modes, but guessing one or two is more usual. As we continue to do this experimentation, I start throwing in questions like "how do we know when we have found all of the normal modes?", or "do you think there might be five, or ten, or a hundred normal modes?", or things like that to try to get them to connect these physical questions with the linear algebra theory that they already know. (Sometimes I even try to cast doubt on their being any normal modes at all, given what they have already seen of the (apparently) complicated motions of the system.) Eventually it (usually) emerges that there can only be three normal modes, because there can only be three linearly independent eigenvectors. Almost every time students will (eventually) guess the eigenvectors $[1, 1, 1]$ and $[1, 0, -1]$, but coming up with the third from pure physical intuition is less common, although not all that rare.

Somewhere around this time I recall that K is real symmetric, and mention that a lot is known about the special eigenvalue/eigenvector properties of such matrices. And then suggest, "Let's see how much further we can get based on this theory". (I think it is always good to keep reminding students of the value of theory, especially engineering students, who tend to be rather sceptical of theory, and prefer to escape to their relative "comfort zone" of concrete numbers and calculations.)

Slide 7 can be used in two ways. One way is as a reminder of theory that they have already seen, and thus as a prompt to try to make use of this theory as a means to *guide* the search for normal modes. Alternatively, if students have not yet encountered these results, to foreshadow the coming of this theory, and by its use in helping to understand this device, to highlight the value of theory.

(15) Six DE solutions: The alert reader will have noticed a slight disconnect between the linear algebra and DE theory here. A 3×3 matrix can have only three linearly independent eigenvectors, but a SECOND-order system of 3 linear DEs requires *six* linearly independent solutions to form a general solution. The form of the normal modes given on transparency 5 resolves this discrepancy. Expanding $\cos(\beta t + \gamma)$ shows that each

eigenvector actually generates two linearly independent solutions of the DE system, one with cos and one with sin. I have never been "called out" on this discrepancy, and do not spontaneously bring it up myself, but it is good to be prepared ahead of time for this question if it arises.

(16) The fact that eigenvectors in this problem DO NOT depend on the physical parameters seemed rather remarkable to me for some time. However, this property is an immediate consequence of the simple observation that the matrix K can be expressed as $aI_3 + bT_3$, where

$$T_3 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix},$$

so that the eigenvectors of K are the same as the eigenvectors of the fixed matrix T_3. This has several nice consequences for the use of this device.

If students are making their own devices at home, or in groups, then this fact ensures the reproducibility of results from one group to another. Each group may not make the "same" device; the mass of the balls, the length of the suspension string, the tension in the "horizontal" cord may all differ from one group to another, but all groups should in principle get the same eigenvectors.

Indeed, this theoretical non-dependence of eigenvectors on physical parameters provides an experimental *opportunity.* Intentionally vary the physical parameters from one group to another (or within one group doing multiple experiments with related devices), to see if this non-dependence actually occurs in practice. And then use this to explore the physical parameter space, to see if the measured frequencies of normal modes (i.e., eigenvalues) vary as predicted by the eigenvalue formulas of the model.

(17) WHY is K symmetric?: I do not always have time to say anything about the "mystery" about why K turns out to be a symmetric matrix. So sometimes it just gets left as a mystery. (For many years, for me it was also a big mystery.) But when I do say anything about it, it is to connect the appearance of symmetric matrices in many applications to a theorem from multivariable calculus, which also appeared to me as a student to be both esoteric, abstract, mysterious, and probably useless. This theorem is sometimes known as *Clairaut's theorem*, or the "equality of mixed second partial derivatives" of (sufficiently smooth) multivariable functions.

Another way to derive a model for this toy shows that the matrix K is the Hessian matrix (of second partial derivatives) of a potential energy function, which by Clairaut's theorem MUST be a symmetric matrix. I think this is the underlying reason for the ubiquity of symmetric matrices in models of so many applications. (Do you think that I am correct in thinking this?)

For many students, it would be good to know that the appearance of symmetric matrices in applications is NOT an accident! And also that Clairaut's theorem is not just a bit of esoterica, but a vitally important property of smooth multivariable functions.

(18) Clairaut counting problem: This note is completely tangential to the main theme of the video, taking off from Clairaut's theorem in a rather different direction. I have several

times used Clairaut's theorem as the source of a nice counting problem. For a real-valued function of n variables, there are n^k symbolically distinct kth partial derivatives. If Clairaut's theorem were not true, then generically one would expect them all also to be distinct as functions. The problem, then, is to count how many of these n^k kth partial derivatives are actually distinct (at least generically) as functions (assuming a sufficient degree of smoothness). In a sophomore level multivariable calculus course, I would pose this as an extra credit problem, with specific (small) concrete values for n and k, e.g., $n = 5$ and $k = 4$, or something like that. This gives students some concrete numerical appreciation for the impact that Clairaut's theorem has on the number of distinct partial derivatives. The problem in full generality is more appropriate for upper-level students, or even first-year graduate students, although it can be used with lower-level students as an indicator of underlying mathematical talent or interest (i.e., potential math majors or minors). I have used it in all of these settings.

(19) Symmetric matrix properties: For some courses, this is just a reminder of stuff that they have already seen in the class. For other courses, this is all completely new information to the students, but information that they are now primed and ready to recognize as useful and relevant. Particularly useful (in the way that I usually do the demo) is the orthogonality of eigenvectors. Once students have identified $[1, 1, 1]$ and $[1, 0, -1]$ as eigenvectors, they can easily check that they are orthogonal (using dot product). And then one can ask the students to produce a third vector that is orthogonal to both of these. Eventually somebody will realize that they can produce such a vector using the cross product, and thus emerges a third eigenvector candidate, which can be checked in the usual two ways. And theory now tells us that we are done; we have three normal modes, and the set of all possible motions of the system (which may have appeared to students to be impossibly complicated based on the first experiments at the beginning of the demo) is now seen to be just all linear combinations of these three special motions. (Sometimes I might even initially encourage this view that the motions are very complicated, in order to eventually be able to highlight the power of mathematical theory to provide insight into that apparent complexity.)

(20) Gerschgorin yet again: Here is another alternative spot where Gerschgorin could be inserted into the discussion, if desired.

(21) Invariant subspaces: Many times when I do this presentation I go from slide 7 directly to slide 8 and conclude there. Slide 8 is mainly a summary of some of the main results from the analysis of the toy, but the fourth bullet does mention one other linear algebra concept that can be physically illustrated using the toy, that is, the notion of invariant subspace. The example in the notes is probably the simplest one to illustrate the notion, i.e., the 2-dim'l invariant subspace spanned by the modes with amplitude patterns $[1, 1, 1]$ and $[-1, 2, -1]$.

Start out the system with initial condition $[0, c, 0]$ for any convenient nonzero c, e.g., start from just the sum of these two eigenvectors. Then watch the system evolve with an amplitude pattern that remains a linear combination of $[1, 1, 1]$ and $[-1, 2, -1]$, albeit with time-varying scalar multipliers in the linear combination. Physically, this just means that the outer balls stay locked together for all time. Note that the other two natural 2-dim'l invariant subspaces observable in this device (combinations of two out of the three eigenvectors) also have some analogous simple physical invariant that is diagnostic of that space. I leave it to the reader to discover those features.

One can also clearly see in the motion of each individual ball in this example the physical phenomenon known to musicians and electrical engineers as "beats". This phenomenon is discussed in pretty much all elementary DE texts as part of the analysis of "forced vibrations". But this toy is a nice example where you can see "beats" without any external forcing at all, just the "internal" interaction of two normal modes with different frequencies. (The wikipedia article on "Acoustic Beats" (actually "Beat (acoustics)") also has some nice illustrations like you would find in a DE text. See Section 5.1 for more on this.)

This 2-dim'l subspace would be called the (sub)space of "symmetric modes" by the chemists and physicists, whereas the 1-dim'l space spanned by the mode associated with $[1, 0, -1]$ is the (sub)space of "anti-symmetric modes". This classification of modes by "symmetry type" is extremely important in analytic chemistry, in particular in infrared and Raman spectroscopy of molecules. In my earlier days with this demo I used to say a good bit more about this, and even had a slide 9 and 10 prepared for this purpose. I almost never use these two slides any more in the classroom, and tend to say very little about molecular spectroscopy and molecular vibrations, except for Math Club talks, or other special audiences. But I will say more about this business of symmetry in later annotations.

(Note that discussing molecular vibrations opens up the possibility of students being able to physically experience eigenvectors in yet another way, beyond just "seeing" an eigenvector. Students can also *dance* an eigenvector! I have used this in the last two years as the conclusion of a Math Club talk, that I have now given three times, with rather good results. This is discussed in the video.)

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NOTE #

(22) Physical symmetry $\rightsquigarrow$ commuting matrices: I added slides $7a$, $7b$, and $7c$ in the early 2000's, for use when I did this demo in a second course in linear algebra for upper-level undergraduates at Kalamazoo College, when I was teaching there. The main point was to show how *physical* symmetry in a system can be used to simplify the analysis of a system, although it rarely is enough by itself to completely determine everything about the dynamics of a system. Symmetry has the effect of putting constraints on the possible dynamics of a system, as well as restricting what can be a possible model for that system.

For the toy, the "obvious" physical symmetry is the reflection symmetry that simply interchanges the outer balls. A novel "proof technique" was used to justify this claim (see slide). I have always wanted to find another use for this proof technique, but so far I have not found any. ;-)

Note that the matrix for this reflection symmetry is just the 3×3 permutation matrix

$$S = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} . \quad (1)$$

In slide 7*b* we see how a dynamical symmetry leads to commuting operators. This was one of the first times I saw a concrete (and simple) example showing how commuting operators could arise naturally from a physical problem. For me, it was good to see that the study of commuting operators was a natural thing to do, and not just an "academic" exercise.

(Discussion continued in next note.)

(23) Consequences of commuting matrices: Perhaps the most basic property of commuting operators is that they share invariant subspaces! In particular, any eigenspace of one must be an invariant subspace of the other. So for our device, the eigenspaces of the simple operator (the permutation matrix) must automatically be invariant subspaces for the toy. For the permutation, there are two eigenvalues (+1 and -1), with a 2-dim'l and 1-dim'l eigenspace, respectively. The 1-dim'l eigenspace of the permutation (with eigenvalue -1) is spanned by $[1, 0, -1]$, which must also (by symmetry considerations ALONE!) necessarily span a 1-dim'l invariant subspace (i.e., an eigenspace) for K. Note that this conclusion is INDEPENDENT of any details of the matrix K! It doesn't matter what the entries of K are at all, this eigenvector is forced on it by the constraints of symmetry. The normal mode for this 1-dim'l subspace is called an "anti-symmetric mode", because it corresponds to the eigenvalue -1 for the reflection symmetry. The modes in the 2-dim'l subspace are called "symmetric modes", because they correspond to the eigenvalue $+1$ for the reflection symmetry.

Finally, in slide 7*c* I noted that all of the symmetries of any fixed system collectively form a group, and that this is how group representation theory finds its way into chemistry and physics. This was meant partly as an encouragement for these students to go on and study abstract algebra, since they could now see that at least part of it has a very physical motivation. For this particular group of students, that included a few physics and chemistry majors, I also included a quotation from a book describing how this entrance of group theory into physics was not met with open arms by all of the physics community. (*Group theory and physics*, S. Sternberg, 1994, see p.x–xi of the preface.)

I have only used this symmetry material a handful of times, since there usually is not enough time to include it in a single session, and often the audience does not have the proper background to catch on to it. (See Section 5.2 and 5.3 for more on this.)

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NOTE #

(24) Explicit formulas: Here are some observations about the dependence (or lack of dependence) of the eigenvectors and eigenvalues on the four physical parameters of the

problem. The fact that it is possible in this problem to see explicitly what these dependencies are, provides a compelling reason *not* to go to numerical values for the parameters "prematurely". Of course it is not always possible to determine these dependencies in every problem, but it is important not to miss the opportunity to find them when it is possible.

In general it is important for students to appreciate the value of this kind of knowledge. For engineers, such information can potentially play a vital role in design problems.

(25) Experiments: The fact that we have explicit expressions for all of the eigenvalues, and hence all of the normal mode frequencies, means that it should be possible to deduce some of the values of the parameters g, ℓ, k, m by instead measuring time. Use a stopwatch to measure all three normal mode periods, and thence get their frequencies. We can now instantly get the ratios $a = g/\ell$ and $b = k/m$.

(We can also see if the three measured frequencies are internally consistent with the predicted formulas $a, a+b$, and $a+3b$ for the eigenvalues. In other words, do the squared frequencies, put in increasing order, have successive differences with the qualitative pattern b, $2b$, and $3b$ for the outer frequencies?)

Knowing these "measured" values for a and b, we can now deduce other values by making some additional measurements. Measure the length ℓ, and deduce g (then compare to known value of g). Or measure m on a scale, and deduce k. This is perhaps the most interesting one, since the coupling constant k is the least accessible to direct measurement. Other variations of these "measurement exercises" can be formulated. Take g as known, and in effect use a stopwatch to "measure" the length ℓ (and then of course one can compare to the directly measured ℓ).

Have two sets of balls available, one set that is significantly heavier than the other set. Then do the frequency measurements with each set, and see if the results are coherent with the predictions of this analysis. And so on ...... I'm sure other interesting variations will occur to you (and maybe independently to the students).

I have never tried any of these things with students, but it might provide an interesting supplement to Raf's practicum. And in general be fun for engineering and physics students, to get their hands on some concrete numbers.

(Continue in next note.)

(26) More experiments: Another natural extension that students might try is to expand the system from 3 to 4 pendula, but still symmetrically arranged to maintain the reflection symmetry. Students should find it a straightforward and completely doable exercise to model this expanded system, getting a 4×4 symmetric (and centrosymmetric!) matrix K with now four (orthogonal) eigenvectors. Can they find them all, by some combination of physical and mathematical techniques? The analysis done earlier with the reflection symmetry will again provide a commuting operator

$$S = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

whose eigenspaces will once more give invariant subspaces for the matrix K. In this case the subspace of "symmetric modes" will still be 2-dim'l, while the "anti-symmetric mode" subspace will increase from 1-dim'l to 2-dim'l. Although I haven't done any experiments or computations, my guess is that there will be two symmetric modes with sign patterns $(+,+,+,+)$ and $(-,+,+,-)$, and two anti-symmetric modes with sign patterns $(+,-,+,-)$ and $(+,+,-,-)$. Students may be able to guess something along these lines, and then compute to confirm or disconfirm. (For more on the 4-pendula and 5-pendula systems see Section 5.2 and Section 5.3.)

(27) Critique of model (a): For this final bit of commentary, let's give some more serious consideration to a critique of the model we have used for the toy. At the most basic level, it should be recalled that one of the first steps in the modeling of this system was to use the linear approximation for the motion of each individual pendulum, that is, the well-known "small-angle" approximation. But we have certainly not been displaying small-angle motions when we were demonstrating the behavior of this system. Since pendulum motion is really a nonlinear phenomenon, to what extent does this call our model (and its predictions) into question? This issue may be beyond the scope of an introductory course, but it is still worth considering, at least in the background. One particular aspect of this issue that probably requires some rethinking is the definition of what a "normal mode" should be. Requiring all particles in the system to be moving in "simple" harmonic motion may be too restrictive. Perhaps instead we should only ask for every particle to be in periodic motion of some kind, with the same waveform (up to a scalar multiple) and the same "phase". Will this revised notion of normal mode still lead (in a nonlinear setting) to a correspondence between "amplitude patterns" and something that can sensibly be labelled as "eigenvectors"?

Another criticism of this model concerns the individual pendula, which are not really standard pendula. Observe that the upper attachment point of each pendulum line is not fixed(!), but moves (albeit not very much) along with the loose (moving!) suspending cable. This effect has not been included in the model. Does neglecting it matter very much? (I don't know.)

(Critiquing commentary continued in next two notes.)

(28) Critique of model (b): But back to our linear model, and trying to make it as plausible as possible. Thinking about the coupling between these parallel pendula, and observing the actual device in action (especially looking from above), it seems quite plausible that the coupling is NOT just between nearest neighbors. There is very likely some non-negligible coupling between ball 1 and ball 3. (Here's one possible experiment to probe this conjecture: remove the middle ball 2 from the device, then set one of the remaining balls in motion.) How will neglecting this smaller "long-range" coupling affect the validity, accuracy, or usefulness of the model and its predictions? One way to address this issue is to just compare the predictions of the model with the observed (and measured) behavior of the device itself.

Another approach is to modify the model and add some terms to account for some longer range coupling (probably relatively small) between balls 1 and 3. This would add some small nonzero entries in the $(1,3)$ and $(3,1)$ positions of K. When we do this it should still produce a new matrix K that commutes with the matrix S of the reflection symmetry, and is still a real symmetric matrix (because of the Hessian argument). Note that a general 3×3 matrix commutes with the reflection matrix S if and only if it is centro-symmetric. So any physically realistic K for this toy must be symmetric and centro-symmetric, hence must have the form

$$\begin{bmatrix} a & b & c \\ b & e & b \\ c & b & a \end{bmatrix}.$$

Adding some terms to the model to account for small coupling between balls 1 and 3 will then introduce one new physical parameter (the long-range coupling constant), and lead to small changes in four entries of the original model. The original zeroes in the $(1,3)$ and $(3,1)$ locations will now become small (but equal) nonzero values (positive or negative, I haven't figured out). And the $(1,1)$ and $(3,3)$ entries will also have some (presumably) small perturbations; again these small changes will be equal, in order that new matrix K remains centro-symmetric. The opportunity is now there to either find new explicit formulas for the eigenvectors and eigenvalues, or (for more advanced students?) to carry out a perturbation analysis of the effect of these new, but small, coupling terms.

(Last part of critiquing commentary continued in next note.)

(29) Critique of model (c): We know, from symmetry considerations discussed in an earlier note, that these small perturbations cannot affect the presence of the eigenvector $[1, 0, -1]$, although it may affect the predicted corresponding eigenvalue. But how will it affect the other two modes? (We know that these two other modes must still be "symmetric" modes, in the sense that they must lie in the $(+1)$-eigenspace of the reflection matrix S, spanned by the vectors $[1, 0, 1]$ and $[0, 1, 0]$.) Perhaps this question will yield to a simple perturbation analysis?

This "enhanced" model, adding one new parameter to account for the long-range coupling, may give a further opportunity for using the measured values of the periods (equivalently, frequencies) of the three normal modes. These three measurements gave more than enough information to determine the ratios $a = g/\ell$ and $b = k/m$. (In fact, redundant information for that purpose.) Perhaps, though, this redundancy of information can be used to tease out numerical information about the long-range coupling constant, too. Or at least to make some estimate of its relative size?

The fact that the modeling of this device is amenable to the kind of critiquing just described may make it particularly well-suited for use in a general course about mathematical modeling. It seems to just naturally raise (and confront students with) many of the general philosophical issues common to the practice of modeling. But it also allows the possibility of being able to try to investigate those issues in a concrete way.

(30) Insight: This is probably Richard Hamming's most famous (and most important) epigram.

5 Appendix

This new appendix expands on two topics mentioned briefly in earlier notes, as well as a new property not previously discussed. The first is the appearance of "beats" in the motion of balls moving in two-dimensional invariant subspaces of the stiffness matrix K, as mentioned in Note # 21. The second is the derivation of explicit formulas for the eigenvalues and eigenvectors of the 4-ball and 5-ball analogs of the system with 3 balls that has been the main focus of these notes. This derivation exploits the physical reflection symmetry of the system in an essential way. The new property concerns the general n-ball system, and how eigenvalues are interlaced as n increases.

5.1 The Beat Phenomenon

When the device moves in a 2-dim'l invariant subspace, then the motion of the balls can exhibit the "beats" phenomenon. (See Note # 21.) Here's an example. Consider the vectors

$$\mathbf{v}_1 := \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \ \mathbf{v}_2 := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{w}_1 := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{w}_2 := \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}.$$

It is easy to see that

$$\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2) \ = \ \operatorname{span}(\mathbf{w}_1, \mathbf{w}_2) \ =: W\,.$$

That W is an invariant subspace of K can be seen in two ways. The simplest is that W is the span of two eigenvectors of K. But an independent reason is that W is the (+1)-eigenspace of the reflection symmetry S in (1) of Note # 22. And since S and K commute, W must be an invariant subspace of K. Note also that W can be characterized as

$$W = \left\{ (x, y, z)^T \in \mathbb{R}^3 \ \text{ with } \ \mathbf{x} = \mathbf{z} \right\}.$$

This view of W is useful for the following experiment/demonstration.

Observe the system behavior with initial condition $\mathbf{w}_1 + \mathbf{w}_2 = \begin{bmatrix} 0 & 3 & 0 \end{bmatrix}$. If you see that the two outer balls stay together in parallel throughout the motion, then that is the hallmark of the motion being confined to W, and you have just "seen" a 2-dim'l invariant subspace with your own eyes. Note that the solution formula for the DE system with this initial condition $\mathbf{w}_1 + \mathbf{w}_2$ is just the (time-varying) linear combination $\mathbf{x}(t) = \mathbf{w}_1 \cos(\omega_1 t) + \mathbf{w}_2 \cos(\omega_2 t)$, which again emphasizes the invariance of the subspace W.

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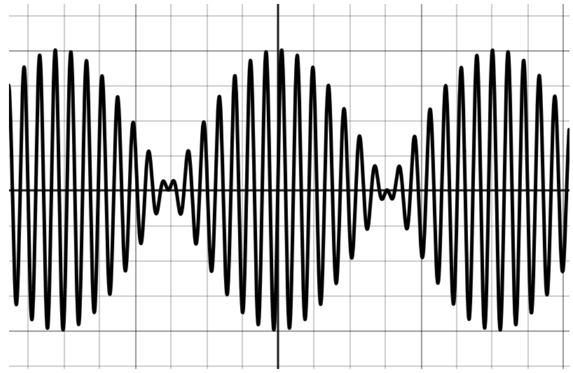
Consider the motion of the (locked-together) outer balls in $\mathbf{x}(t)$. The model predicts that both balls will move as $\cos(\omega_1 t) - \cos(\omega_2 t)$, which can be rewritten using the trig identity

$$\cos(a) - \cos(b) \ = \ 2 \sin\left(\tfrac{b-a}{2}\right) * \sin\left(\tfrac{a+b}{2}\right)$$

into a form which can be viewed as an oscillation of frequency $\frac{1}{2}(\omega_1 + \omega_2)$, but with a *time-varying* amplitude, indeed a sinusoidally-varying amplitude of (lower) frequency $\frac{1}{2}(\omega_2 - \omega_1)$, as in the following sketch.



This kind of signal can be perceived acoustically as a pure tone with sinusoidally-varying volume, an auditory phenomenon often going by the name **"beats"**.

The second ball in this $\mathbf{x}(t)$ (with initial condition $\begin{bmatrix} 0 & 3 & 0 \end{bmatrix}$) undergoes a somewhat more complicated **"beat-like"** motion, given by the formula $\mathbf{x}_2(t) = \cos(\omega_1 t) + 2\cos(\omega_2 t)$. This only-slightly-more-complicated-looking linear combination of sinusoids can be rewritten in the definitely-more-complicated-looking product form

$$\mathbf{x}_2(t) \;=\; A(t)\cos(\alpha_1 t + \phi) \;=\; \sqrt{5 + 4\cos(2\delta_1 t)} * \cos(\alpha_1 t + \phi)\,,$$

where $\alpha_1 = \frac{\omega_1+\omega_2}{2}$, $\delta_1 = \frac{\omega_1-\omega_2}{2}$, and $\phi = \arctan\big[-\frac{1}{3}\tan(\delta_1 t)\big]$ can be viewed as a time-varying phase. Note that the time-varying amplitude $A(t) = \sqrt{5+4\cos(2\delta_1 t)}$ oscillates with lower frequency $2\delta_1 = \omega_1 - \omega_2$ in the interval $1 \le A \le 3$, not between 0 and 3 as for "classical" beats.

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Aside: It is curious that it seems to be *much* more difficult to express a *general linear combination* of two sinusoids in product form, as compared to the relative simplicity of the very special linear combinations $\cos(a) \pm \cos(b)$ and $\sin(a) \pm \sin(b)$. Although somewhat complicated, there is a way to rewrite a general linear combination of two sinusoids (with possibly different amplitudes, frequencies, and phases) as a single product that gives some insight into the behavior of that sum function. Note that this more complicated formula reduces to the better-known trig identities for $\cos(a) \pm \cos(b)$ and $\sin(a) \pm \sin(b)$ when amplitudes are equal. Consider the general linear combination

$$A_1 \sin(\omega_1 t + \psi_1) + A_2 \sin(\omega_2 t + \psi_2)\,. \tag{2}$$

Note that without loss of generality we can assume that A_1 and A_2 are both nonnegative, since any negative signs can be absorbed into the inside of the sin expressions. Now letting $\alpha_1 = \frac{\omega_1+\omega_2}{2}$, $\alpha_2 = \frac{\psi_1+\psi_2}{2}$, $\delta_1 = \frac{\omega_1-\omega_2}{2}$, and $\delta_2 = \frac{\psi_1-\psi_2}{2}$, the linear combination (2) can be rewritten in terms of $\alpha_1, \alpha_2, \delta_1$ and δ_2 to give

$$A_1 \sin[(\alpha_1 t + \alpha_2) + (\delta_1 t + \delta_2)] + A_2 \sin[(\alpha_1 t + \alpha_2) - (\delta_1 t + \delta_2)]\,,$$

which can then be expanded and rearranged to give

$$(A_1 + A_2)[\sin(\alpha_1 t + \alpha_2)\cos(\delta_1 t + \delta_2)] + (A_1 - A_2)[\cos(\alpha_1 t + \alpha_2)\sin(\delta_1 t + \delta_2)]\,.$$

Factoring out $\big[(A_1 + A_2)\cos(\delta_1 t + \delta_2)\big]$ now yields

$$\big[(A_1 + A_2)\cos(\delta_1 t + \delta_2)\big] * \Big[_{\bigstar} \sin(\alpha_1 t + \alpha_2) + \left[\frac{(A_1 - A_2)}{(A_1 + A_2)}\tan(\delta_1 t + \delta_2)\right]\cos(\alpha_1 t + \alpha_2)\Big]_{\bigstar} .$$

Applying the identity $\sin(a) + k\cos(a) = \sqrt{1+k^2}\,\sin\big(a + \arctan(k)\big)$ to the expression in brackets delimited by $\bigstar$'s with $a = (\alpha_1 t + \alpha_2)$ and $k = \frac{(A_1 - A_2)}{(A_1 + A_2)}\tan(\delta_1 t + \delta_2)$, we finally get

$$A_1 \sin(\omega_1 t + \psi_1) + A_2 \sin(\omega_2 t + \psi_2) \;=\; \widetilde{A}(t)\sin((\alpha_1 t + \alpha_2 + \phi)\,,$$

where $\widetilde{A}(t)$ and ϕ are defined by $\phi := \arctan\left[\frac{(A_1 - A_2)}{(A_1 + A_2)}\tan(\delta_1 t + \delta_2)\right]$ and

$$\begin{aligned} \widetilde{A}(t) &:= \sqrt{[(A_1 + A_2)\cos(\delta_1 t + \delta_2)]^2 + [(A_1 - A_2)\sin(\delta_1 t + \delta_2)]^2} \\ &= \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos 2[\delta_1 t + \delta_2]}\,. \end{aligned}$$

The alert reader will have noticed the problematic nature of the formula for the time-varying phase ϕ, due to the singularities in the tangent function. However, a function that is continuous for all $t \in \mathbb{R}$ can be recovered from this formula by judiciously switching branches of arctan when passing through the tangent's singular points.

5.2 4 Balls

Consider the analogous system with 4 balls, with "stiffness" matrix now of the form $K_4 = aI_4 + bT_4$, where T_4 is the 4×4 matrix

$$T_4 \;=\; \begin{bmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 1 \end{bmatrix}.$$

Clearly the eigenvectors of K_4 and T_4 are identical, and the eigenvalues are simply related; if γ is an eigenvalue of T_4, then $a + \gamma b$ is the corresponding eigenvalue of K_4. Thus it suffices to analyze the matrix T_4, which (like K_4) is centrosymmetric, and thus commutes with the reflection symmetry

$$R_4 \;=\; \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

R_4 has two eigenvalues $\lambda_1 = +1$ and $\lambda_2 = -1$, each with a 2-dim'l eigenspace. The R_4-eigenspace for λ_1 is $\mathcal{X} = \operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2\}$, while the eigenspace for λ_2 is $\mathcal{Y} = \operatorname{span}\{\mathbf{y}_1, \mathbf{y}_2\}$, where

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}.$$

Since T_4 and R_4 commute, the R_4-eigenspaces $\mathcal{X}$ and $\mathcal{Y}$ must be invariant subspaces for T_4. Thus the restriction operators $T_4\big|_{\mathcal{X}}$ and $T_4\big|_{\mathcal{Y}}$ are well-defined, and their 2×2 matrix representations are easily found. Consider first the operator $T_4\big|_{\mathcal{X}}$. The given spanning set for $\mathcal{X}$ comprises two orthogonal vectors with equal norm, so the matrix representation of $T_4\big|_{\mathcal{X}}$ with respect to this basis will also be symmetric (although we can no longer expect it to necessarily be centrosymmetric). We find that $T_4\big|_{\mathcal{X}}(\mathbf{x}_1) = \mathbf{x}_1 - \mathbf{x}_2$ and $T_4\big|_{\mathcal{X}}(\mathbf{x}_2) = -\mathbf{x}_1 + \mathbf{x}_2$, so that the matrix representation of $T_4\big|_{\mathcal{X}}$ is just

$$S = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Similarly, a matrix representation of $T_4\big|_{\mathcal{Y}}$ can be found using the given basis of two orthogonal, equal norm vectors in $\mathcal{Y}$. The result is $T_4\big|_{\mathcal{Y}}(\mathbf{y}_1) = \mathbf{y}_1 - \mathbf{y}_2$ and $T_4\big|_{\mathcal{Y}}(\mathbf{y}_2) = -\mathbf{y}_1 + 3\mathbf{y}_2$, giving the matrix representation

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}.$$

The eigenstuff of the 2×2 matrices S and A is now easily found: eigenvalues 0 and 2 for S with corresponding (orthogonal) eigenvectors $[1, 1]^T$ and $[1, -1]^T$; eigenvalues $2 - \sqrt{2}$ and $2 + \sqrt{2}$ for A with corresponding (orthogonal) eigenvectors $[1 + \sqrt{2}, 1]^T$ and $[1, -1 - \sqrt{2}\,]^T$. These finally lead to four orthogonal eigenvectors for T_4 (and hence also K_4). They are displayed below grouped together by *symmetry type*, a notion commonly used by chemists and physicists. Eigenvectors of K_4 (and their corresponding normal modes) that lie in the $+1$-eigenspace of the reflection symmetry R_4 are referred to as *symmetric modes*, while those lying in the -1-eigenspace of R_4 are the *antisymmetric modes*. Here they are, labelled by the corresponding eigenvalue γ of T_4. (Keep in mind that their corresponding eigenvalues of K_4 are of the form $a + \gamma b$.)

Symmetric modes

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$\gamma = 0$ $\qquad$ $\gamma = 2$

Anti-symmetric modes

$$\begin{bmatrix} 1+\sqrt{2} \\ 1 \\ -1 \\ -1-\sqrt{2} \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1-\sqrt{2} \\ 1+\sqrt{2} \\ -1 \end{bmatrix}$$

$\gamma = 2 - \sqrt{2}$ $\qquad$ $\gamma = 2 + \sqrt{2}$

Note that the numerically ordered eigenvalues of T_4 are $0 < (2 - \sqrt{2}\,) < 2 < (2 + \sqrt{2}\,) < 4$. (The last number 4 in this ordered list is included because it is the Gerschgorin upper bound for eigenvalues of T_4, indeed of any T_n. The Gerschgorin lower bound 0 is also an eigenvalue for every T_n.) Here are some further comments on this ordered list of eigenvalues:

- Observe that as we scan through the eigenvalues from smallest to largest, the symmetry type of the corresponding eigenvectors alternates between symmetric and antisymmetric. Is this just a coincidence, or is it happening because of some underlying structural reason? If so, what is that reason, and does it persist for systems with a larger number of balls?

- Another interesting observation comes from comparing the eigenvalues of T_3 with those of T_4 (hence also those of K_3 with those of K_4). We have

$$0 < (2-\sqrt{2}) < \mathbf{1} < \mathbf{2} < \mathbf{3} < (2+\sqrt{2}).$$

The eigenvalue 0 is in common, but the nonzero eigenvalues (those of T_3 in red and those of T_4 in blue) are **interlaced**. Is this also a coincidence, or a more general phenomenon? (See Section 5.3 and 5.4 for more on this.)

5.3 5 Balls

Moving up to the 5-ball system, the "stiffness" matrix is now $K_5 = aI_5 + bT_5$, where T_5 is the 5×5 matrix

$$T_5 = \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}.$$

An analysis similar to the one used for the 4-ball system in Section 5.2 can be used to find explicit expressions for the eigenstuff of the 5-ball system. Note that the number $\phi := \frac{1}{2}(1+\sqrt{5})$, aka the "golden ratio" (not the magenta ratio), will appear repeatedly in this analysis.

Once again, the fact that the matrix T_5 is centrosymmetric, and thus commutes with the reverse identity R_5, can be systematically exploited to reduce the 5×5 eigenproblem to a pair of 2×2 problems, i.e., to restriction operators acting on 2-dim'l invariant subspaces. Begin by observing that R_5 again has the two eigenvalues ± 1, with a 3-dim'l eigenspace for $\lambda = +1$, and a 2-dim'l eigenspace for $\lambda = -1$. These eigenspaces are easily described using the vectors

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} 0 \\ 0 \\ \sqrt{2} \\ 0 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{z}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{z}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix};$$

the $(+1)$-eigenspace is $\mathcal{W} = \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$, while the (-1)-eigenspace is $\mathcal{Z} = \operatorname{span}\{\mathbf{z}_1, \mathbf{z}_2\}$. (Note that the $\sqrt{2}$ in $\mathbf{w}_3$ is chosen so that the basis of vectors for $\mathcal{W}$ consists of orthogonal vectors, *all of the same norm.*)

The restriction operator $T_5\big|_{\mathcal{Z}}$ satisfies $T_5\big|_{\mathcal{Z}}(\mathbf{z}_1) = \mathbf{z}_1 - \mathbf{z}_2$ and $T_5\big|_{\mathcal{Z}}(\mathbf{z}_2) = -\mathbf{z}_1 + 2\mathbf{z}_2$, so the matrix representation of $T_5\big|_{\mathcal{Z}}$ with respect to this basis is

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

Eigenvalues of A are easily found to be $2-\phi$ and $1+\phi$, with corresponding (orthogonal) eigenvectors $[\phi, 1]^T$ and $[1, -\phi]^T$, which then lead to the anti-symmetric modes $\phi\mathbf{z}_1 + \mathbf{z}_2$ and

$\mathbf{z}_1 - \phi\mathbf{z}_2$ displayed below.

Analyzing the restriction operator $T_5\big|_{\mathcal{W}}$ is a bit more involved, since the invariant subspace $\mathcal{W}$ now has dimension 3 rather than just 2. A first attempt would be to find the 3×3 matrix representation of the "full" restricted operator $T_5\big|_{\mathcal{W}}$. Here is what you would get if you use the basis $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ for $\mathcal{W}$: $T_5\big|_{\mathcal{W}}(\mathbf{w}_1) = \mathbf{w}_1 - \mathbf{w}_2$; $T_5\big|_{\mathcal{W}}(\mathbf{w}_2) = -\mathbf{w}_1 + 2\mathbf{w}_2 - \sqrt{2}\mathbf{w}_3$; $T_5\big|_{\mathcal{W}}(\mathbf{w}_3) = -\sqrt{2}\mathbf{w}_2 + 2\mathbf{w}_3$. This gives us the (symmetric!) matrix representation

$$S = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -\sqrt{2} \\ 0 & -\sqrt{2} & 2 \end{bmatrix}$$

for $T_5\big|_{\mathcal{W}}$. One could now go ahead and find the eigenstuff for this S, and thence deduce the symmetric modes for T_5. However, we can do something a little sneakier, and try to exploit the fact that we already know one of the eigenvectors of T_5 without any numerical effort. The vector $\widetilde{\mathbf{w}} = [\,1,\ 1,\ 1,\ 1,\ 1\,]^T \in \mathcal{W}$ is that eigenvector of T_5 (with eigenvalue 0), which can be immediately seen either from the fact that the row sums of T_5 are all zero, or from the physical intuition that $[\,1,\ 1,\ 1,\ 1,\ 1\,]^T$ "ought to give" a normal mode of the 5-ball system. (Note that $\widetilde{\mathbf{w}}$ corresponds to the eigenvector $[\,1,\ 1,\ (\sqrt{2})^{-1}\,]^T \in \mathbb{R}^3$ for S, although it will be convenient to instead use the multiple $\mathbf{v}_1 := [\,\sqrt{2},\ \sqrt{2},\ 1\,]^T$.)

There are two (related) ways that one might exploit the knowledge of this eigenvector. Both involve using the orthogonal complement of the known eigenvector, either $\widetilde{\mathbf{w}}^\perp$ **in** $\mathcal{W}$ for $T_5\big|_{\mathcal{W}}$, or $\mathbf{v}_1^\perp$ for S in $\mathbb{R}^3$, as a *two-dimensional* invariant subspace, and then finding the 2×2 matrix representation of this (even further) restricted operator. Here are the results for $\mathbf{v}_1^\perp \in \mathbb{R}^3$, and the corresponding restriction of S. (Details for working with $\widetilde{\mathbf{w}}^\perp \subset \mathcal{W}$ are left to the reader.) To get an orthogonal basis for $\mathbf{v}_1^\perp$, first note that $\widehat{\mathbf{v}}_2 = [\,1,\ -1,\ 0\,]^T$ is clearly in $\mathbf{v}_1^\perp$. The cross product $\mathbf{v}_1 \times \widehat{\mathbf{v}}_2$ then gives a second vector $\mathbf{v}_3 := [\,1,\ 1,\ -\sqrt{8}\,]^T$ in $\mathbf{v}_1^\perp$. The multiple $\mathbf{v}_2 := \sqrt{5}\,\widehat{\mathbf{v}}_2$ has the same norm as $\mathbf{v}_3$, so we now take $\mathcal{B} = \{\mathbf{v}_2, \mathbf{v}_3\}$ as our orthogonal basis for $\mathbf{v}_1^\perp$. We have $S\big|_{\mathbf{v}_1^\perp}(\mathbf{v}_2) = \frac{1}{2}(5\mathbf{v}_2 - \sqrt{5}\,\mathbf{v}_3)$ and $S\big|_{\mathbf{v}_1^\perp}(\mathbf{v}_3) = \frac{1}{2}(-\sqrt{5}\,\mathbf{v}_2 + 5\mathbf{v}_3)$, so with respect to the basis $\mathcal{B}$, the (symmetric) matrix representation of $S\big|_{\mathbf{v}_1^\perp}$ is

$$\widetilde{S} = \tfrac{\sqrt{5}}{2} \begin{bmatrix} \sqrt{5} & -1 \\ -1 & \sqrt{5} \end{bmatrix}.$$

Note that $\widetilde{S}$ is again centrosymmetric (is this a coincidence?), and so commutes with the 2×2 backwards identity $R_2 = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Consequently, the (1-dim'l) eigenspaces of R_2, spanned by $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$, respectively, must also be eigenspaces for $\widetilde{S}$. Hence the corresponding vectors in $\mathbf{v}_1^\perp \subset \mathbb{R}^3$, i.e.,

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \leftrightarrow \mathbf{v}_2 + \mathbf{v}_3 \qquad \text{and} \qquad \begin{bmatrix} 1 \\ -1 \end{bmatrix} \leftrightarrow \mathbf{v}_2 - \mathbf{v}_3\,,$$

will be eigenvectors for S. Chasing these back into the T_5-invariant subspace $\mathcal{W} \subset \mathbb{R}^5$ gives the remaining eigenvectors of T_5, and hence the symmetric modes of K_5, displayed below. Keep in mind that the labelled γ values are eigenvalues of T_5, while the corresponding eigenvalues of K_5 are $a + \gamma b$.

Symmetric modes **Anti-symmetric modes**

$$\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix},\quad \begin{bmatrix}1+\phi\\-1\\-2\phi\\-1\\1+\phi\end{bmatrix},\quad \begin{bmatrix}-1\\1+\phi\\-2\phi\\1+\phi\\-1\end{bmatrix} \qquad\qquad \begin{bmatrix}\phi\\1\\0\\-1\\-\phi\end{bmatrix},\quad \begin{bmatrix}1\\-\phi\\0\\\phi\\-1\end{bmatrix}$$

$$\gamma = 0 \qquad \gamma = 3-\phi \qquad \gamma = 2+\phi \qquad\qquad \gamma = 2-\phi \qquad \gamma = 1+\phi$$

Note that the numerically ordered eigenvalues of T_5 are

$$0 \;<\; (2-\phi) \;<\; (3-\phi) \;<\; (1+\phi) \;<\; (2+\phi) \;<\; 4\,.$$

- Observe that (just as for T_4) as we scan through the eigenvalues from smallest to largest, the symmetry type of the corresponding eigenvectors alternates between symmetric and antisymmetric. Since this is happening again (and the same is true for T_3) this is starting to look like it is *not* a coincidence. So once again the question arises, even more forcefully, is it happening because of some underlying structural reason? If so, what is that reason, and does it persist for systems with an even larger number of balls?

- Let's again compare eigenvalues of successively-sized T_n, this time those of T_4 with those of T_5 (hence also those of K_4 with those of K_5). We have

  $$0 \;<\; 2-\phi \;<\; (2-\sqrt{2}\,) \;<\; 3-\phi \;<\; \mathbf{2} \;<\; 1+\phi \;<\; (2+\sqrt{2}\,) \;<\; 2+\phi\,.$$

  The eigenvalue 0 is in common, but the nonzero eigenvalues (those of T_5 in red and those of T_4 in blue) are **interlaced**. Is this also a coincidence, or a more general phenomenon? (See Section 5.4 for more on this.)

5.4 n Balls and Eigenvalue Interlacing

Consider the $n \times n$ version of the stiffness matrix, i.e., $K_n := aI_n + bT_n$, where

$$T_n \;=\; \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}_{n\times n},$$

and denote the eigenvalues of K_n by

$$\lambda_1(K_n) \le \lambda_2(K_n) \le \cdots \le \lambda_n(K_n)\,.$$

Then it is easy to see that $\lambda_1(K_n) = a$ for all n, equivalently, that $\lambda_1(T_n) = 0$ for all n, since all row sums of every T_n are zero. However, I claim that all of the other eigenvalues of K_n and K_{n+1} are **interlaced**, in the sense that

$$\lambda_2(K_{n+1}) < \lambda_2(K_n) < \lambda_3(K_{n+1}) < \lambda_3(K_n) < \cdots < \lambda_n(K_{n+1}) < \lambda_n(K_n) < \lambda_{n+1}(K_{n+1})\,. \quad (3)$$

Since $b > 0$, to show this interlacing it suffices to show that the corresponding interlacing holds for the eigenvalues of T_n. Note that this claim is consistent with the earlier concrete results for $n = 3$ and $n = 4$ in Sections 5.2 and 5.3.

To justify the claim, observe that there is a simple way to build T_{n+1} from T_n. First border T_n with an extra row and column of zeroes to get

$$\widetilde{T}_{n+1} = \begin{bmatrix} 1 & -1 & & & & 0 \\ -1 & 2 & -1 & & & 0 \\ & \ddots & \ddots & \ddots & & \vdots \\ & & -1 & 2 & -1 & 0 \\ & & & -1 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}_{(n+1)\times(n+1)},$$

Note that the eigenvalues of $\widetilde{T}_{n+1}$ are exactly the same as those of T_n, except for the addition of a second zero eigenvalue, i.e., zero is an eigenvalue of $\widetilde{T}_{n+1}$ with multiplicity 2. We can now recover T_{n+1} as a *rank-one update* of $\widetilde{T}_{n+1}$ by adding the rank-one matrix $\left[\begin{smallmatrix} +1 & -1 \\ -1 & +1 \end{smallmatrix}\right]$ to the lower right-hand corner of $\widetilde{T}_{n+1}$. The interlacing claim now follows from Corollary 4.3.9 on p.241 of Horn & Johnson's *Matrix Analysis* (2nd Ed.).

- A final question: is it possible to use this construction of T_{n+1} from T_n as a basis for developing a general and explicit closed-form formula for any part of the eigenstuff of T_n, and hence also of K_n? (Perhaps it is more reasonable to hope for some kind of recursive formula, rather than something completely explicit and closed-form.)