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"SWINGING BALLS DEMO"

$\boxed{\text{Took about 45 min}}$ (John Fink's Linear Algebra Class, Wed 25 Nov 1998)

- Explain System
  Demonstrate variety / complexity of motions

Also, given in a slightly modified version to Nil's MTH 510 Lin Alg. (graduate) class at WMU that same week

- Develop model (quickly)

  (1) Simple pendulum

  Solutions
  $\ddot{x} + kx = 0$ where $k = \frac{g}{\ell} > 0$
  $x(t) = A\cos(\sqrt{k}\cdot t) + B\sin(\sqrt{k}\cdot t)$
  Set of all solns. is a (2)-dim'l vector space
  [All linear combinations of 2 "basic" solutions]

Simplifying assumptions
(a) No friction
(b) Small-angle approx.
(c) Spring model is valid for our system.

  (2) Coupled pendula (with springs)

  Coupling Forces:
  Elongation of (first) spring $= x_2 - x_1$
  ~~Magnitude of~~ Spring force $= \pm k(x_2 - x_1)$
  ⋮

  [Hooke's law – magnitude of spring force is proportional to its elongation/compression from its "resting" length]

  $\boxed{\text{CLAIM}}$ This is also a reasonable model for the swinging balls!

  (3) Matrix Version

  $$\boxed{\ddot{x} = -\underset{\sim}{K}\,x} \quad \text{with} \quad \underset{\sim}{K} = \begin{pmatrix} a+b & -b & 0 \\ -b & a+2b & -b \\ 0 & -b & a+b \end{pmatrix} \qquad a = \frac{g}{\ell},\ b = \frac{k}{m}$$

  "Linear system of DEs"

  [NOTE: Gerschgorin implies posdef with eigenvalues between a and $a+4b$.]

Same freq. AND "in phase"

- A Very Special Kinds of motion – (4) "NORMAL MODES"

  Can you "guess" any normal modes?

  It is not obvious (at least to me) that there should be any normal modes at all!!!
  And if there are any, how many there should be!

- Connection to Linear Algebra

  (5), (6) ⟵ Need positive eigenvalues

  Simplifications
  (No phase shifts, No cosines)

  (Gerschgorin can show that this is what we actually get!)

Back to:

- Normal Modes of the "Swinging Balls" $\longleftrightarrow$ Eigenpairs of $K = \begin{pmatrix} a+b & -b & 0 \\ -b & a+2b & -b \\ 0 & -b & a+b \end{pmatrix}$

  Check: $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector.

  Others?

  Eigenvectors don't depend on the values of a and b (physical significance)

- K is a real symmetric matrix (not an ACCIDENT)

  It's the Hessian (matrix of 2^{nd} derivs) of a potential energy function.

  [Equality of mixed partials $\Rightarrow$ Symmetry of matrix]

  (7) Basic Facts about real symmetrics

  [Beautiful Theory and Important Applications]

- Use physical intuition, (physical) symmetry, orthogonality of e-vectors, etc to find all normal modes/eigenvectors.

  [Gerschgorin $\Rightarrow$ Posdef with all e-values in interval $[a, a+4b]$!]

- (8) CONCLUSIONS

  (1) Exactly 3 normal modes. WHY? 3x3 matrix has at most 3 lin. indep. eigenvectors.

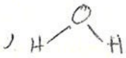
  (2) {Motions} = {Linear combin's of normal modes} !!

  despite the "apparent complexity" of the system motions.

  (3) "(Physical) Independence of Modes" – Start system in one mode, never see any character of the others.

  (4) Invariant Subspaces EX: $\underbrace{\begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}}_{\text{Initial Condition}} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \in \text{span}\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}\right)$

  [Outer Balls will stay together for all time]

REMARKS: Vibrational Spectroscopy of Molecules (Infrared / Raman) Ex: O=C=O,  (Find more Modes?)

END

Back to:

- $\boxed{\text{Normal Modes of the "Swinging Balls"}}$ $\longleftrightarrow$ Eigenpairs of $\mathbb{K} = \begin{pmatrix} a+b & -b & 0 \\ -b & a+2b & -b \\ 0 & -b & a+b \end{pmatrix}$

<u>Check</u>: $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector.

Others?

Eigenvectors <u>don't</u> depend on the values of a and b (physical significance)

- $\mathbb{K}$ is a <u>real</u> <u>symmetric</u> matrix (<u>not</u> an ACCIDENT)

It's the <u>Hessian</u> (matrix of 2^{nd} derivs) of a potential energy function.

[Equality of mixed partials ⇒ Symmetry of matrix]

7 Basic Facts about real symmetrics

[Beautiful Theory <u>and</u> Important Applications]

- <u>Second</u> - 7a Can use <u>physical</u> <u>symmetry</u> of the system to tell us <u>more</u> about the <u>possible</u> e-vectors / invariant subspaces.

- SLOGAN -
"(Phys.) Symmetry leads to Commuting Operators" (!)

7a

...tion, (physical) symmetry, orthogonality of e-vectors, <u>etc</u>.

... normal modes / eigenvectors.

[Gerschgorin ⇒ Posdef with all e-values in interval $[a, a+4b]$!]

... 3 normal modes. <u>WHY</u>? (3x3 matrix has at most 3 lin. indep. eigenvectors.)

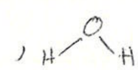
2 {Motions} = {<u>Linear combin's</u> of normal modes} !!
despite the "<u>apparent complexity</u>" of the system motions.

3 "(Physical) Independence of Modes" - Start system in one mode, never see any character of the others.

4 Invariant Subspaces <u>EX</u>: $\begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \in \text{span}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$

Initial Condition

[Outer Balls will stay together for all time]

<u>REMARKS</u>: Vibrational Spectroscopy of Molecules <u>EX</u>: O=C=O , 
(Infrared / Raman) (Find more Modes?)

END

~~NOTE: Eigenvectors (normal mode amplitude patterns) don't depend on the ~~actual~~ (particular) values of the physical parameters g, l, k, m at all. Just on the "structure" of the problem. An important observation that would have been obscured by jumping to numerical values of g, l, k, m prematurely.~~

~~But the eigenvalues (equiv. normal mode frequencies) do depend on the numerical values of g, l, k, m. Still, it is useful not to jump too quickly to particular values, since the exact way in which the eigenvalues depend on the parameters will be hidden and not be transparent.~~

~~MORAL: Symbolic computation is still of crucial importance, even with the availability of powerful number crunching.~~

~~Hamming - "The purpose of computing is insight, not numbers"~~

NOTE: The eigenvalues (normal mode frequencies) depend on the physical parameters g, l, k, m, but the eigenvectors (normal mode amp. patterns) don't! They seem to depend only on the "structure" of the problem (i.e. equal masses, evenly-spaced, equal length pendula, equal springs). This is an important observation that would have been obscured by jumping (too soon) prematurely to particular numerical values for g, l, k, m. Even for the eigenvalues, it is useful not to jump too quickly to numerical values for the physical parameters, since the manner (the precise way) in which the eigenvalues depend on the parameters will be hidden (and not transparent).

MORAL: Even with the availability of powerful number crunching, symbolic computation and the patterns it can reveal is still of crucial importance.

Hamming - "The purpose of computing is insight, not numbers"
(analysis, mathematics, modelling, science ...)

Find out more about the normal modes of CO_2 and H_2O. Specific stuff - like the shapes and frequencies!!
