Exciting Eigenvectors: Seeing is Believing

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ILAS-Madrid, 12 June 2023

At a Pub in Galway

Introducing the “Toy”

- How to construct ...
- Some sample motions ...
- Notation ... (position vector)

Introducing the "Toy"

- How to construct ...

- Some sample motions ...

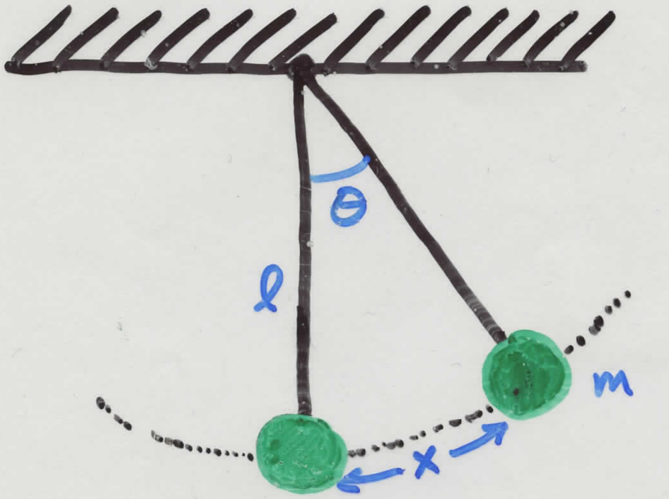
- Notation ... (position vector)

$$\mathbf{x}(t) \;:=\; \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \;\in\; \mathbb{R}^3$$

My Old Transparencies

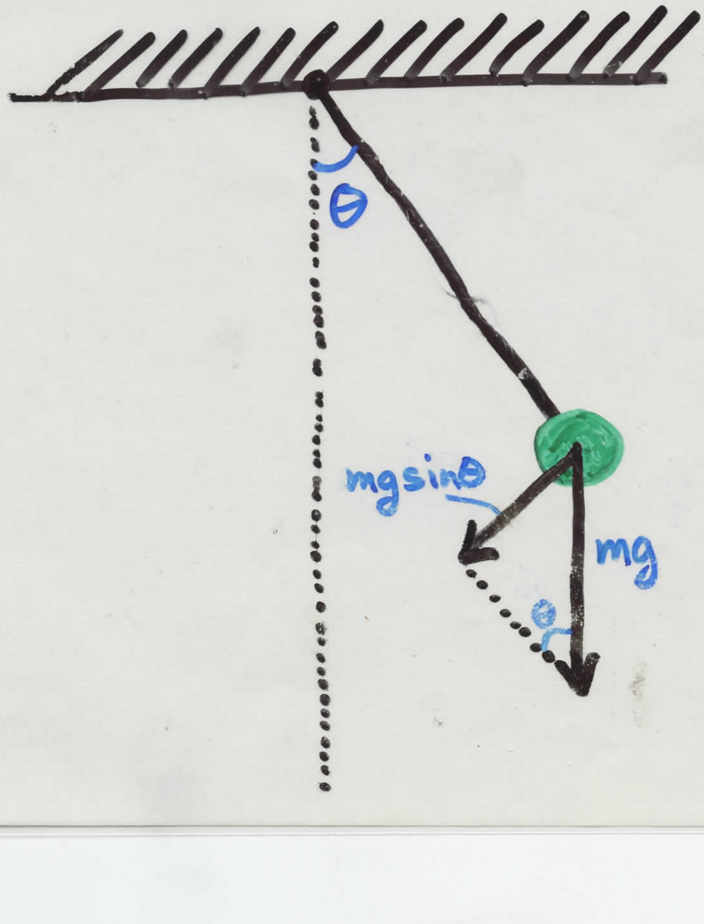
For a bit of nostalgia .... (from the 1990's)

PENDULUM



$$\theta = \frac{x}{l}$$

Force diagram:



$$ma = F = -mg\sin\theta$$

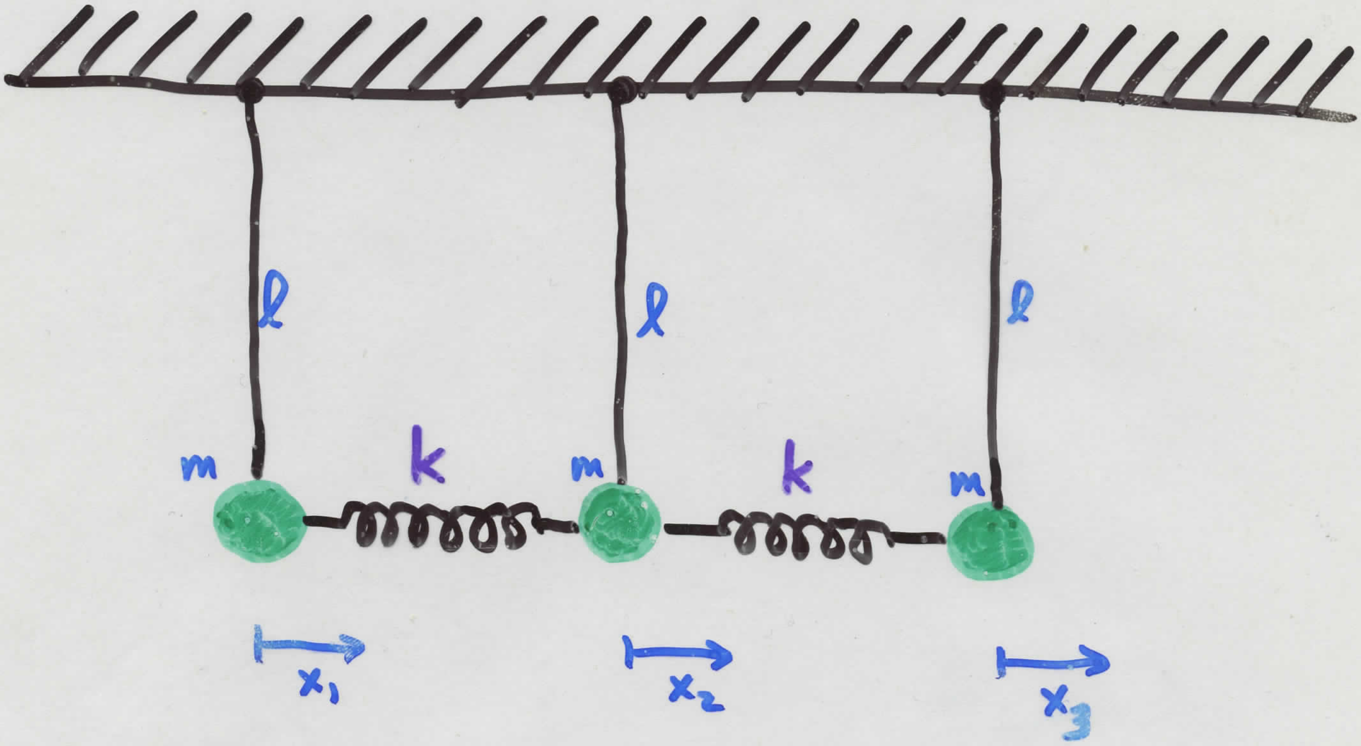
$$m\ddot{x} = -mg\sin\theta$$

$$m\ddot{x} \approx -mg\theta$$

$$m\ddot{x} \approx -mg\left(\frac{x}{l}\right)$$

$$\boxed{m\ddot{x} \approx \frac{-mg}{l}x}$$

COUPLED PENDULA



$$m\ddot{x}_1 = -\frac{mg}{l}x_1 + k(x_2 - x_1)$$

$$m\ddot{x}_2 = -k(x_2 - x_1) - \frac{mg}{l}x_2 + k(x_3 - x_2)$$

$$m\ddot{x}_3 = -k(x_3 - x_2) - \frac{mg}{l}x_3$$

" ◯ " - Spring Forces
[the <u>coupling</u>]

MATRIX VERSION

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} = -\begin{pmatrix} \frac{g}{\ell}+\frac{k}{m} & -\frac{k}{m} & 0 \\ -\frac{k}{m} & \frac{g}{\ell}+\frac{2k}{m} & -\frac{k}{m} \\ 0 & -\frac{k}{m} & \frac{g}{\ell}+\frac{k}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\boxed{\ddot{x} = -Kx}$$

$$K = \begin{pmatrix} a+b & -b & 0 \\ -b & a+2b & -b \\ 0 & -b & a+b \end{pmatrix}$$

$$a = \frac{g}{\ell} \qquad b = \frac{k}{m}$$

Why is the matrix K symmetric?

Is it just a **coincidence**? Or is it happening for a reason?

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Come back to this later.

And there is some **additional significant structure** in the matrix K.

Can you identify it?

CLAIM

The linear 2^{nd}-order system of ODEs $\boxed{\mathbf{x}'' = -K\mathbf{x}}$ with

$$K = \begin{bmatrix} a+b & -b & 0 \\ -b & a+2b & -b \\ 0 & -b & a+b \end{bmatrix} \quad \text{and} \quad a = \frac{g}{\ell}\,,\; b = \frac{k}{m}$$

is also a **reasonable model** for the coupled pendula "toy".

Note: engineers often refer to K (without the m's)
as the **stiffness matrix**.

NORMAL MODES

A normal mode of a system is a pattern of motion where all the components (particles) are moving in <u>simple harmonic motion</u> with the <u>same frequency</u> of oscillation and the <u>same phase</u>, although <u>not</u> necessarily with the same amplitude.

Normal modes show up in all kinds of vibrating systems - from molecules to bells to bridges + buildings.

Connection to Linear Algebra

Normal Mode in mathematical language:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} a_1 \cos(\beta t + \gamma) \\ a_2 \cos(\beta t + \gamma) \\ a_3 \cos(\beta t + \gamma) \end{pmatrix} = \underbrace{\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}}_{\text{Vector of constants}} \cdot \underbrace{\cos(\beta t + \gamma)}_{\text{Time-varying scalar}}$$

"Amplitude Pattern"

$$\boxed{x(t) = \mathbf{a} \cdot \cos(\beta t + \gamma)}$$

For $x(t)$ to be a possible motion, it must satisfy the equation $\ddot{x} = -\mathbf{K}x$.

- $\ddot{x} = \mathbf{a} \cdot \frac{d^2}{dt^2}\left[\cos(\beta t + \gamma)\right] = -\beta^2 \mathbf{a} \cdot \cos(\beta t + \gamma)$
- $-\mathbf{K}x = -\mathbf{K}\mathbf{a}\cos(\beta t + \gamma)$

these must be = for all times t !!

$$\mathbb{K}\mathbf{a}\cos(\beta t+\gamma) = \beta^2\mathbf{a}\cos(\beta t+\gamma)$$

$$(\mathbb{K}\mathbf{a}-\beta^2\mathbf{a})\cos(\beta t+\gamma) \equiv 0$$

$$\Longrightarrow \quad \boxed{\mathbb{K}\mathbf{a} = \beta^2\mathbf{a}} \, .$$

Normal Mode	$\longleftrightarrow$	Eigenvector/Eigenvalue Pair for the matrix $\mathbb{K}$

Eigenvector = "Amplitude Pattern"

Eigenvalue = $(\text{Frequency})^2 > 0$

Seeing some Eigenvectors?

Now let's try to **excite** some eigenvectors! **"Eigenmodes"**

Seeing some Eigenvectors?

Now let's try to **excite** some eigenvectors! **"Eigenmodes"**

And maybe **theory** can help guide us to these eigenvectors.

REAL SYMMETRIC MATRICES

- All eigenvalues are real.
- There is always a basis made up solely of eigenvectors.
- The eigenvectors in this basis can always be chosen to be mutually orthogonal.

<u>REMARK</u>: None of these properties are true for matrices in general.

Ways to find Eigenvectors

- Guess .... (take advantage of theory)

  and "check" .... (physically for normal mode, **"Eigenmode"**
  and/or mathematically by multiplication Kx)

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- Textbook method:
  - Find **characteristic polynomial** for $K = \begin{bmatrix} a+b & -b & 0 \\ -b & a+2b & -b \\ 0 & -b & a+b \end{bmatrix}$

  $$\begin{aligned} p(\lambda) &= \det(\lambda I - K) \\ &= \lambda^3 - (3a+4b)\lambda^2 + (3a^2+8ab+3b^2)\lambda - (a^3+4a^2b+3ab^2) \end{aligned}$$

    (Here $a = \frac{g}{\ell}$ and $b = \frac{k}{m}$.)

  - **Roots** of $p(\lambda)$ are the eigenvalues of K.
    Find them (as functions of the physical parameters a and b).

  - For each root (eigenvalue) λ, **solve** $Kx = \lambda x$ for an eigenvector x.
    (Surprise!? Eigenvectors are **independent** of the parameters!)

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Short answer: **Clairaut's Thm** (from Calc III) "Clairaut counting problem"

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<u>Longer answer</u>: K is the **Hessian matrix** Hf of a potential energy function f.

Recall that for $f(x, y, z)$,

$$Hf := \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix}.$$

Relationships between these 2^{nd} partial derivatives?

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More details: Newton's law

$$\begin{aligned} M\mathbf{x}'' = \mathbf{F} &= -\big(\boldsymbol{\nabla} f\big)(\mathbf{x}) \\ &= -\boldsymbol{\nabla}\Big(f(0,0,0) + \nabla f|_{(0,0,0)} \cdot \mathbf{x} + \frac{1}{2}\mathbf{x}^T\big(Hf|_{(0,0,0)}\big)\,\mathbf{x} + \text{ h.o.t.}\Big) \\ &\approx -\big(Hf|_{(0,0,0)}\big)\,\mathbf{x} = -K\,\mathbf{x} \end{aligned}$$

SYMMETRY Leads to COMMUTING Operators

- <u>Symmetry</u> of a System
  - "Static" System

    An invertible transformation (permutation of particles) that leaves system "unchanged"

    <u>EX:</u>

    

    Reflection

    

    Rotation

    .... etc.

  - "Dynamic" System (where the motions of system are the main object of study)

    Permutation of material particles that maps

    $$\left\{ \begin{matrix} \text{Possible} \\ \text{motions} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{Possible} \\ \text{motions} \end{matrix} \right\}$$

- <u>CLAIM</u>

  Interchange of the outer two balls is a symmetry of the "Coupled Pendulums" system.

  <u>Proof</u>

  Walk around to the other side. $\square$

- **More Mathematically**

If S is a symmetry, then

$\vec{x}(t)$ a motion of system (i.e. a solution of DE) $\Rightarrow$ $S\vec{x}(t)$ is also a possible motion, i.e. $\vec{y}(t) \underset{def}{=} S\vec{x}(t)$ is also a solution of DE

In other words,

$$\ddot{x}(t) = -K x(t) \Rightarrow \ddot{y}(t) = -K y(t)$$
$$\Rightarrow [Sx(t)]'' = -KSx(t)$$
$$\Rightarrow S\ddot{x}(t) = -KSx(t)$$
$$\Rightarrow S[-Kx(t)] = -KS\,x(t)$$
$$\not\Rightarrow \qquad \vdots \qquad \forall x(t)!!$$
$$\not\Rightarrow$$
$$\Rightarrow \boxed{SK = KS} \; . \; \square$$

- Any symmetry S must commute with K, and conversely!
- Check: $S = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. $SK \overset{?}{=} KS$.
- By analyzing the simple operator S, we automatically learn about the more complicated operator K.

Additional Structure of K

Commuting with the matrix $S = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ is the same as **centrosymmetry**!

Check it out in general, and specifically for

$$K = \begin{bmatrix} a+b & -b & 0 \\ -b & a+2b & -b \\ 0 & -b & a+b \end{bmatrix}.$$

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$$K = \begin{bmatrix} a+b & -b & 0 \\ -b & a+2b & -b \\ 0 & -b & a+b \end{bmatrix}.$$

Consequence: any eigenspace of S is an invariant subspace for K.

With $\mathbf{v} := \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T$, $\mathrm{span}(\mathbf{v})$ is a 1-dim'l eigenspace of S.

Hence $\mathbf{v}$ **must be** an eigenvector of K, regardless of the values of the physical parameters a and b, just due to the physical reflection symmetry of the device.

LINEAR ALGEBRA and GROUP THEORY

- The more symmetries of a system we have to work with, the more we can learn about $\mathbb{K}$. Each extra symmetry puts more constraints on what the behavior of the system could possibly be.

- In general, the set of all symmetries of a system (static or dynamic) forms a group, and this is how group theory enters into physics and chemistry. [EXERCISE ....]

- More specifically it is group representation theory, at the crossroads of group theory and linear algebra, that is a crucial mathematical element of modern physics. Here one studies abstract groups via their realizations as groups of linear operators.

- ASIDE: Group theoretical methods were NOT initially welcomed with open arms by all physicists.

  "Gruppenpest"

The explicit recognition of the importance of group representation theory in physics started very soon after the discovery of quantum mechanics, with the path-breaking work of Weyl, Wigner, and others. In fact, Weyl's classic book of 1928, *Gruppentheorie und Quantenmechanik*, makes instructive and inspiring reading even today. (In his book, Weyl adopts the pedagogic strategy of segregating the mathematics and the physics into separate chapters. There is much to be said for this strategy, especially from the point of view of logical coherence. But it had the unintended effect that physicists and mathematicians would read alternate chapters. I have taken the risk of going to the opposite extreme here, trying to use the physics to motivate the mathematics and vice versa, mixing the two.) The uses of group theory in quantum mechanics extended from chemistry and spectroscopy in the 1920s and 1930s, to nuclear and particle physics in the 1930s and 1940s, and then to high energy physics and the discovery of the theory of colored quarks in the 1960s and 1970s. It is this story of the interweaving of mathematics and physics that I try to tell in this book.

It should not be supposed that there was a warm reception in the physics community to the introduction of group theoretical methods. In fact, the contrary was true. To get a feeling for a typical early reaction, let me quote at length from the autobiography of John Slater, who was a leading American physicist and head of the MIT Physics Department for many years. The following quotes are taken from pages 60–2 of his autobiography:

> It was at this point that Wigner, Hund, Heitler, and Weyl entered the picture with their "Gruppenpest": the pest of the group theory.... The authors of the "Gruppenpest" wrote papers which were incomprehensible to those like me who had not studied group theory, in which they applied these theoretical results to the study of the many electron problem. The practical consequences appeared to be negligible, but everyone felt that to be in the mainstream one had to learn about it. Yet there were no good texts from which one could learn group theory. It was a frustrating experience, worthy of the name of a pest.
>
> I had what I can only describe as a feeling of outrage at the turn which the subject had taken....
>
> As soon as this [Slater's] paper became known, it was obvious that a great many other physicists were as disgusted as I had been with the group-theoretical approach to the problem. As I heard later, there were remarks made such as "Slater has slain the 'Gruppenpest'". I believe that no other piece of work I have done was so universally popular.

Outrage, disgust, the characterization of group theory as a plague or as a dragon to be slain – this is not an atypical physicist's reaction in the 1930s–50s to the use of group theory in physics. It is, however, amazing to consider that this autobiography was published in 1975, after the major triumphs of group theory in elementary particle physics.

From: S. Sternberg – "Group Theory and Physics" [1994]

CONCLUSIONS

- There are exactly three normal modes for this system of coupled pendula.
  [REASON - ? ? ?]
- Every possible motion of this system is a linear combination of normal modes.
- "Independence of Modes"
- Invariant Subspaces

REMARKS on Vibration of Molecules,
Infrared Spectroscopy,
and Group Theory.

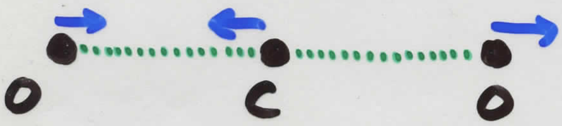
Can you "DANCE" an Eigenvector?

Are you willing to give it a try?

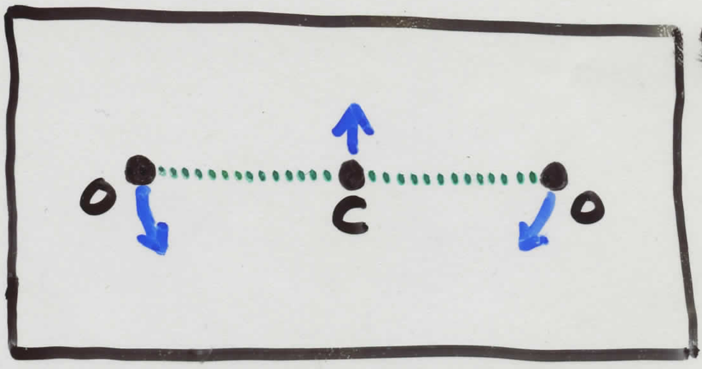
Molecular Vibrations

$\boxed{CO_2}$ has <u>four</u> vibrational normal modes

"Stretching Modes"

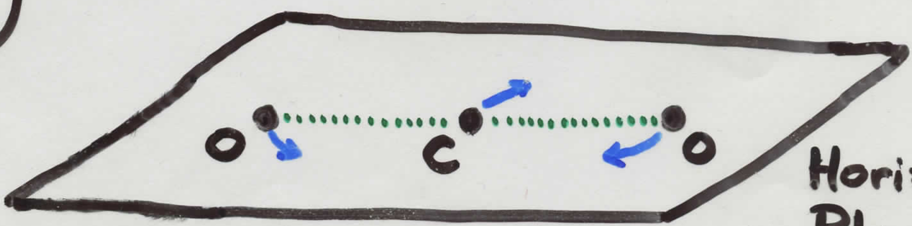


"Bending Modes"



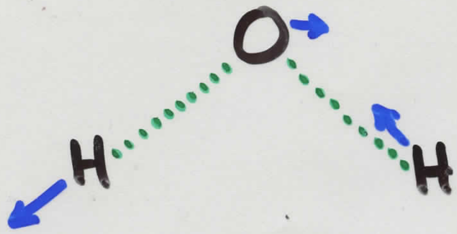
Vert. Plane

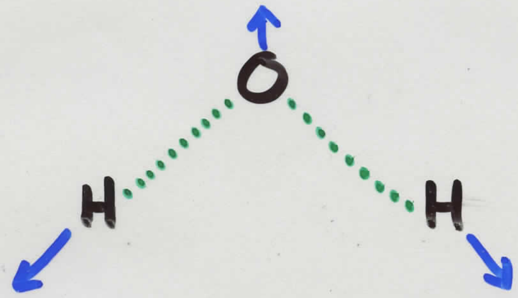
(Doubly Degenerate)

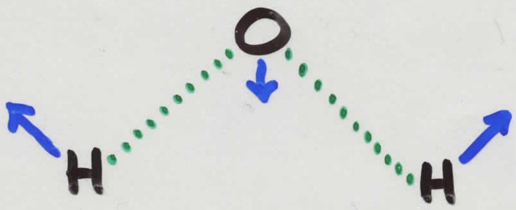


Horiz. Plane

$\boxed{H_2O}$ has <u>three</u> vibrational modes







Our Hope

- Now that you have **seen** 3 eigenvectors and **danced** 7 more, we hope that some of you will try using this toy in your own courses.

- Even better, that you will find good ways for students to work directly with this device (e.g., in small groups, or even individually at home!), in a more **exploratory and interactive** way.

- Many possible **extensions** can be explored ....
  - More balls ... (effect on model, and its predictions)
  - Physical experiments ... (use stopwatch to measure **"eigenfrequencies"**, and deduce k, ℓ, m, g?)
  - Second set of 3 identical balls, but with different mass ...
  - Many others ...

Some Resources

<u>Available on the ILAS-Education website</u>:

`https://la-education.oucreate.com`

- These talk slides
- Annotated handwritten notes of DSM
- **H.V. McIntosh (1952)** – ????????????????????????

  Matrix Analysis II: Further Introduction and
  Some Applications to Physical Problems
  (mimeograph book)

- Raf's video podcast from 5 Apr 2022

  "Integrating Applications of Linear Algebra Across the Curriculum"

  `https://la-education.oucreate.com/monthly-seminar-abstract-for-april-5-2022/`

MATRIX ANALYSIS II

FURTHER INTRODUCTION AND SOME APPLICATIONS TO PHYSICAL PROBLEMS

1952

H. V. McINTOSH

Table of Contents for Part II

$$Z_{eff} = \frac{V_1 \cosh n\varphi + Z_0 I_1 \sinh n\varphi}{\frac{V_1}{Z_0} \sinh n\varphi + I_1 \cosh n\varphi}$$

$$\lim_{n\to\infty} Z_{eff} = \frac{V_1 e^{n\varphi} + Z_0 I_1 e^{n\varphi}}{\frac{V_1}{Z_0} e^{n\varphi} + I_1 e^{n\varphi}} \quad \left(e^{-n\varphi} \doteq 0\right)$$

$$= Z_0 = \sqrt{\frac{r}{g}} \frac{1}{\sqrt{1+\frac{rg}{4}}}$$

If, as is the case in practice, both r and g are small, The second radical is approximately 1, giving

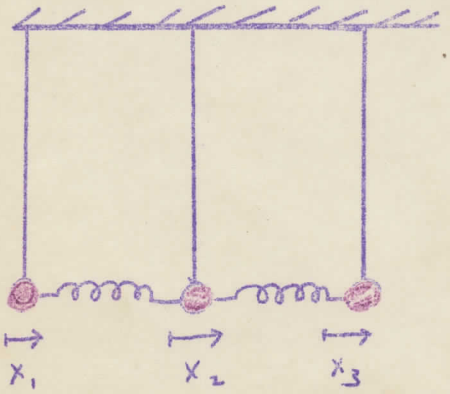
$$Z_{eff} \doteq \sqrt{r/g}$$

which is the geometric mean between the resistance of the spans and the resistance to ground. To choose some numbers, if $r = 0.1$ ohm, $g = 10^{-6}$ ohm, the resistance of the long line would be about 300 ohms.

THE COUPLED HARMONIC OSCILLATOR

The expansion of the elastic constant for a vibrational problem according to Sylvester's theorem has its interpretation in terms of normal modes. If a certain displacement is an eigenvector for the elastic constant, the restoring force will always be opposite in direction to the displacement, and proportional to it. The resulting motion is a particularly simple type of motion since then all the particles will vibrate with simple harmonic motion, and the "shape" of the motion, so to speak, will remain constant and only its amplitude will vary with time. Such motion is called a normal mode. The situation is frequently described by saying the "time is separable," meaning that the solution to the problem may be written as a product of two factors, one depending only upon the time, and the other depending only upon the coordinates of the particles.

The displacements corresponding to eigenvectors can then be discovered by noticing which displacements of the particles give rise to forces always pushing back directly along the line of displacement. As an example consider three pendula whose bobs are all of the mase mass, m, and the lengths, ℓ, of whose suspensions are all the same; but which are, however, interconnected by springs of elastic constant k, as illustrated in the diagram. Considering



only plane motion with small enough amplitude to justify the approximation $\theta \doteq \sin\theta$ the forces are:

$$f_1 = -kx_1 + kx_2 - \frac{mg}{\ell}x_1$$

$$f_2 = kx_1 - 2kx_2 + kx_3 - \frac{mg}{\ell}x_2$$

$$f_3 = kx_2 - kx_3 - \frac{mg}{\ell}x_3$$

Equating the restoring faces to the inertia forces, and writing the equations in matrix form:

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} = - \begin{bmatrix} \frac{k}{m}+\frac{g}{\ell} & -\frac{k}{m} & 0 \\ -\frac{k}{m} & \frac{2k}{m}+\frac{g}{\ell} & -\frac{k}{m} \\ 0 & -\frac{k}{m} & \frac{k}{m}+\frac{g}{\ell} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

which has the form

$$\ddot{X} = -K X \qquad 1)$$

The eigenvalues of K are the solutions of the equation

$$\begin{vmatrix} \frac{k}{m}+\frac{g}{\ell}-\lambda & -\frac{k}{m} & 0 \\ -\frac{k}{m} & \frac{2k}{m}+\frac{g}{\ell}-\lambda & -\frac{k}{m} \\ 0 & -\frac{k}{m} & \frac{k}{m}+\frac{g}{\ell}-\lambda \end{vmatrix} = \left(\frac{k}{m}+\frac{g}{\ell}-\lambda\right)^2\left(\frac{2k}{m}+\frac{g}{\ell}-\lambda\right) - \frac{2k^2}{m^2}\left(\frac{k}{m}+\frac{g}{\ell}-\lambda\right) = 0$$

and are

$$\lambda_1 = \frac{g}{l}$$

$$\lambda_2 = \frac{g}{l} + \frac{k}{m}$$

$$\lambda_3 = \frac{g}{l} + \frac{3k}{m}$$

The eigenvectors can be calculated from the formula:

$$|i\rangle\langle i| = \frac{\prod_{j\neq i}(K - \lambda_j \mathbb{I})}{\prod_{j\neq i}(\lambda_i - \lambda_j)}$$

$$|1\rangle\langle 1| = \frac{(K - \lambda_2 \mathbb{I})(K - \lambda_3 \mathbb{I})}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}$$

$$= \frac{1}{(-\frac{k}{m})(-\frac{3k}{m})} \begin{bmatrix} 0 & -\frac{k}{m} & 0 \\ -\frac{k}{m} & \frac{k}{m} & -\frac{k}{m} \\ 0 & -\frac{k}{m} & 0 \end{bmatrix} \begin{bmatrix} -\frac{2k}{m} & -\frac{k}{m} & 0 \\ -\frac{k}{m} & -\frac{k}{m} & -\frac{k}{m} \\ 0 & -\frac{k}{m} & -\frac{2k}{m} \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \therefore \quad \boxed{|1\rangle = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$$

Likewise

$$|2\rangle\langle 2| = \frac{(K - \lambda_1 \mathbb{I})(K - \lambda_2 \mathbb{I})}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)}$$

$$= \frac{1}{\frac{k}{m}(-\frac{2k}{m})} \begin{bmatrix} -\frac{k}{m} & -\frac{k}{m} & 0 \\ -\frac{k}{m} & \frac{2k}{m} & -\frac{k}{m} \\ 0 & -\frac{k}{m} & \frac{k}{m} \end{bmatrix} \begin{bmatrix} -\frac{2k}{m} & -\frac{k}{m} & 0 \\ -\frac{k}{m} & -\frac{k}{m} & -\frac{k}{m} \\ 0 & -\frac{k}{m} & -\frac{2k}{m} \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad \boxed{|2\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}$$

and

$$|3\rangle\langle 3| = \frac{(K-\lambda_1 \mathbb{I})(K-\lambda_2 \mathbb{I})}{(\lambda_3-\lambda)(\lambda_3-\lambda_2)}$$

$$= \frac{1}{\frac{3k}{m}\ \frac{2d}{m}} \begin{bmatrix} \frac{k}{m} & -\frac{k}{m} & 0 \\ -\frac{k}{m} & \frac{2k}{m} & -\frac{k}{m} \\ 0 & -\frac{k}{m} & \frac{k}{m} \end{bmatrix} \begin{bmatrix} 0 & -\frac{k}{m} & 0 \\ -\frac{k}{m} & \frac{k}{m} & -\frac{k}{m} \\ 0 & -\frac{k}{m} & 0 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\therefore \quad |3\rangle = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Therefore the matrix

$$U = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$$

diagonalizes K and

$$U^{-1}KU = \begin{bmatrix} g/l & 0 & 0 \\ 0 & g/l + k/m & 0 \\ 0 & 0 & g/l + 3k/m \end{bmatrix}$$

Under the substitution $U^{-1}X = Y$, equation 1) reads

$$\ddot{Y} = -U^{-1}KUY$$

which gives the following set of differential equations when the corresponding elements are equated.

$$\ddot{y}_1 = -\frac{g}{l} y_1$$

$$\ddot{y}_2 = -(g/l + k/m) y_2$$

$$\ddot{y}_3 = -(g/l + 3k/m) y_3$$

which have as their solutions

$$y_1 = a_1 \sin\left(\sqrt{g/l}\, t + \delta_1\right)$$

$$y_2 = -a_2 \sin\left(\sqrt{g/l + k/m}\, t + \delta_2\right)$$

$$y_3 = a_3 \sin\left(\sqrt{g/l + 3k/m}\, t + \delta_3\right)$$

So that by making the unsubstitution $X = UY$

$$X = |1\rangle y_1 + |2\rangle y_2 + |3\rangle y_3$$

and one obtains the positions of the individual pendula as a function of time according to

$$x_i = \langle i:1\rangle y_1 + \langle i:2\rangle y_2 + \langle i:3\rangle y_3$$

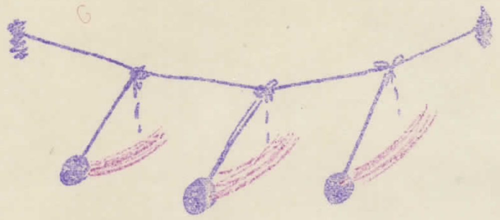
The quantities $y_i = \langle i|X\rangle$ are called normal coordinates, while motion described by just one of them is a normal mode. There is, of course, a geometrical interpretation of the normal coordinate as projections of the eigenvectors on the coordinate axes, but since it is difficult to see the connection of such a space directly with the motion of the particle, this interpretation does not have a direct physical meaning.

A slightly modified problem, which however has the same equation of motion as the one discussed consists in three identical pendula hung from a string which has been rather loosely stretched between two supports. The coupling between the pendula which swing in a plane perpendicular to the supporting string is provided through this support rather than by the connecting springs contemplated in the problem.

The normal mode of lowest frequency has the eigenvector

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

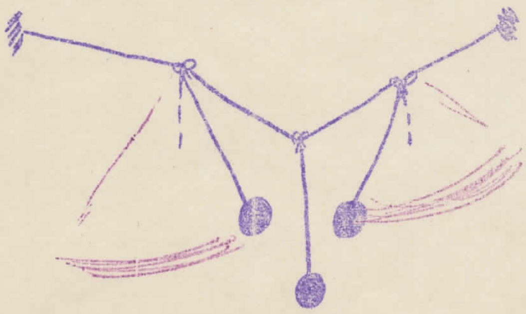
so that when it is excited all the pendula have the same displacement of any given time. Thus there is no force on the supports between pendula, and all three swing as a unit. Since there is no additional force they swing with the frequency that they would have when uncoupled; namely $\nu = \frac{1}{2\pi}\sqrt{g/l}$



The second eigenvector is

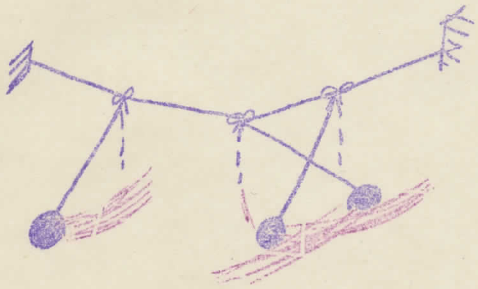
$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

and describes motion in which the center hob is at rest and the other two are displaced oppositely, but by the same amount. There is no resultant force on the center hob, since the influences of the side hobs cancel, being equal, but oppositely directed. Nevertheless the end pendula manage to exert a force on one another such that one acts to retard the other. The action is symmetric between the end hobs, and altogether the condition for an eigenvector is seen to be fulfilled. The increased restoring force on the end hobs raises their frequency from the uncoupled $\nu = \frac{1}{2\pi}\sqrt{g/l}$ to $\frac{1}{2\pi}\sqrt{g/l + k/m}$.



The third eigenvector is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

so that in this case the two end hobs move together but with the center hob always displaced twice as far in the opposite direction. The frequency of this motion is still higher than that of the other normal modes due to the greater retaining faces exerted due to the coupling, and becomes $\frac{1}{2\pi}\sqrt{\frac{\ell}{g}+\frac{3k}{m}}$. Again inspection shows that the forces are always opposite and proportional to the displacements, so that the condition for eigenvectors of the elastic constant is again fulfilled.

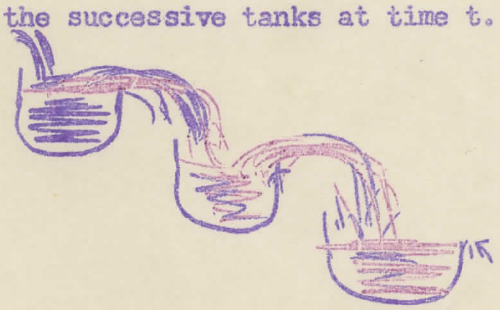


The fact that the motion is described in terms of eigenvectors of the elastic constant shows that motion originally started according to one normal mode can nveer proceed according to another. A rthaer striking example of this independence is afforded by considering the motion resulting after the center hob has been displaced and then released, the end hobs starting from rest. This is readily seen to be a combination of motion of the types 1 and 3, 2 being absent. Type 2 provides the motion in which the end hobs move in opposite direction, so it follows that from these initial conditions one will never expect to see the two end hobs going in opposite direction, a conclusion borne out by calculation and experiment. Actually the center hob will move with gradually increasing amplitude, finally coming to rest when the outer hobs have taken up the motion, which in their turn begin to lose amplitude, setting the center hob to swinging once more, and so on.

A ROW OF TANKS

Suppose the first tank in a row contains a mixture of G-g gallons of liquid A and G gallons of liquid B. From time $t = 0$, liquid A is pumped into the first tank at the rate of r gallons per minute and a mixture of A and B is thus forced from each tank to the next at the same rate. Assuming that the tanks are perfectly stirred, calculate the amount of liquid B in any tank as a function of the time.

Such a problem as this helps to illustrate the nature of a diffusion process. Suppose that $x_n(t)$ $n = 0, 1, ---$ denotes the amount of liquid B in the successive tanks at time t.



In a time Δt sec,

$$r \frac{x_{k-1}}{G} \Delta t$$

gallons of B flow into tank k, while

$$r \frac{x_k}{G} \Delta t$$

gallons flow out. Then

$$\Delta x_k = \frac{r \Delta t}{G}(x_{k-1} - x_k)$$

gives the change of liquid B in the k th tank in Δt sec. Thus the rate of flow of B is

$$\frac{dx_k}{dt} = \frac{r}{G}(x_{k-1} - x_k)$$

and setting $x_{-1} = 0$ will give correctly $\frac{dx_0}{dt}$. The equation above is and element of a matric equation.